

# A PREDATOR-PREY MODEL WITH STRONG ALLEE EFFECT IN THE PREY AND AN ALTERNATIVE FOOD SOURCE FOR THE PREDATOR

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**Abstract—** In this paper, we have studied the dynamical behaviors of a predator-prey model, where the prey is subject to strong Allee effect, and the predator has an alternative food source. Positivity and boundedness of the system are discussed. Some extinction criteria are derived. Stability analysis of the equilibrium points is presented including some global results. Numerical simulations are carried out to validate our analytical findings. Implications of our analytical and numerical findings are discussed critically.

**Key words:** Predator-prey, Allee effect, extinction, stability.

## I. INTRODUCTION

Nowadays the study of predator-prey relationship has become an extremely important part of ecology. In this relationship, one species serves as a food for the other. It is true that the preys always try to develop the methods of evasion to avoid being eaten. However, it is certainly not true that a predator-prey relationship is always harmful for the preys, it might be beneficial to both. Further, such a relationship often plays an important role to keep ecological balance in nature.

Mathematical modelling of predator-prey interaction was started in 1920s. Interestingly, the first predator-prey

model in the history of theoretical ecology was developed independently by Alfred James Lotka (a US physical chemist) and Vito Volterra (an Italian mathematician) [25, 42]. Subsequently, this model has been used as a machine to introduce numerous mathematical and practical concepts in theoretical ecology. Many refinements of the Lotka-Volterra model have also been made to overcome the shortcomings of the model and to get better insights of predator-prey interactions. In the last five or six decades, a number of predator-prey models are developed and systematically cultured in literature. However, urge for incorporating many parameters of real systems had been felt day by day.

If we summarize the basic considerations behind the modelling of predator-prey systems, it would be evident that the most crucial elements of predator-prey models are the choices of *growth function* of the prey and *functional response* of the predator.

It has long been recognized that the famous *logistic growth function* has the capability of describing individual population growth. The function is introduced in 1838 by the Belgian mathematician Pierre Francois Verhulst [41] and later it is rediscovered in 1920 by American biologists Reymon Pearl and Lowell Reed [33]. If  $X(T)$  denotes the population density at time  $T$ , then the logistic growth equation is given by

$$\frac{dX}{dT} = rX \left( 1 - \frac{X}{K} \right), \quad (1.1)$$

$$\frac{dX}{dT} = rX \left( 1 - \frac{X}{K} \right) \left( \frac{X}{K_0} - 1 \right), \quad (1.2)$$

here  $r$  is the intrinsic per capita growth rate and  $K$  is the carrying capacity of the environment. The logic behind this is very simple. As the resources (e.g., space, food, and essential nutrients) are limited, every population grows into a saturated phase from which it cannot grow further; the ecological habitat of the population can carry just so much of it and no more. This suggests that the per capita growth rate is a decreasing function of the size of the population, and reaches zero as the population achieved a size  $K$  (in the saturated phase). Further, any population reaching a size that is above this value will experience a negative growth rate. The term  $-rX^2 / K$  may also be regarded as the loss due to intraspecific competition. Although logistic growth function became extremely popular, but, in real life situations, researchers found many evidences where the populations show a reverse trend in low population density [14, 12, 30, 9, 15, 36]. This phenomenon of positive density dependence of population growth at low densities is known as the *Allee effect* [39, 15].

The phenomenon of *Allee effect* is named after the US Behavioral scientist Warder Clyde Allee (although Allee never used the term 'Allee effect'). Allee described this concept in three of his papers [3, 4, 5]. Actually, the term 'Allee effect' was introduced by Odum [32]. Since the late eighties of the 20th century, the concept gained importance but there were necessity of clear-cut definitions and clarification of concepts.

The necessity was fulfilled when three reviews by Stephens et al. [39], Courchamp et al. [14], Stephens and Sutherland [38]. There are many reasons for Allee effect, such as difficulty in mate finding, reduced antipredator vigilance, problem of environmental conditioning, reduced defense against predators, and many others (for thorough reviews, see references [9, 15]). The Allee effect can be divided into two main types, depending on how strong the per capita growth rate is depleted at low population densities. These two types are called the strong Allee effect [40, 44, 45] or critical depensation [10, 11, 23], and the weak Allee effect [39, 43] or noncritical depensation [10, 11, 23]. Usually, the Allee effect is modelled by a growth equation of the form

Where  $X(T)$  denotes the population density at time  $T$ ,  $r$  is the intrinsic per capita growth rate, and  $K$  is the carrying capacity of the environment. Here  $0 < K_0 \ll K$ . When  $K_0 > 0$  and the population size is below the threshold level  $K_0$ , then the population growth rate decreases [6, 13, 18, 22], and the population goes to extinction. In this case, the equation describes the *strong Allee effect* [40, 44, 45]. On the contrary, the description of *weak Allee effect* is also available (see references [44, 19]). In this paper, we are concerned with strong Allee effect. The above growth is often said to have a *multiplicative Allee effect*. There is another mathematical form of the growth function featuring the *additive Allee effect*. In this paper, we are not interested in additive Allee effect (interested readers might see the works of Aguirre et al. [1, 2]). A comparison of the logistic growth function of (1.1) and the function representing Allee effect in equation (1.2) can be found in [29].

Let us now turn our attention from the individual growth of the prey to the interaction of the prey and its predator. The function that describes the number of prey consumed per predator per unit time for given quantities of prey and predator is known as the *functional response* or *trophic function*. Depending upon the behaviour of populations, more suitable functional responses have been developed as a quantification of the relative responsiveness of the predation rate to change in prey density at various populations of prey. In this connection, Holling family of functional responses are the most focused [20, 21]. The Holling type-I functional response (or the Lotka-Volterra functional response) is given by  $F(X) = \beta X$ , where  $X(T)$  is the prey density at time  $T$  and  $\beta > 0$  is a constant. The Holling type-II (or Michaelis-Menten) functional response has become extremely popular. The type-II functional response includes the fact that a single individual can feed only until the stomach is not full, and so a saturation function would be better to describe the intake of food. This is similar to the concept of the *law of diminishing returns* borrowed from operations research, via a hyperbola rising up to an asymptotic value. In other words, the functional response would be of the following form

$$F(X) = \frac{\beta X}{\alpha + X} \quad (1.3)$$

Where  $X(T)$  is the prey density at time  $T$ ,  $\beta > 0$  is the maximal growth rate of the predator and  $\alpha > 0$  is the half saturation constant. Although these functional responses have served as basis for a very large literature in predator-prey theory (see [31, 37, 27], and references therein); but there should be no denying that, in many situations, the predator density could have a direct effect on functional response. Interestingly, J.R. Beddington[7] and D.L.DeAngelis (and his co-researchers) [16] independently introduced a 'predator- dependent' functional response in the same year 1975. This functional response is known as *Beddington-DeAngelis functional response*, which is given by

$$F(X) = \frac{\beta X}{\alpha + \gamma X + \delta Y} \quad (1.4)$$

Here the term  $\delta Y$  measures the mutual interference between predators.

Once again, we want to go back to the question of individual population growth. Even if a population is not susceptible to Allee effect, it is easy to understand that the logistic growth function might not be capable of explaining the growth of each population. Obviously, several modifications of logistic growth function have been proposed, whilst retaining the sigmoid and asymptotic property of the logistic curve. In 1959, Richards [35] suggested the growth equation in the following form:

$$\frac{dX}{dT} = rX \left[ 1 - \left( \frac{X}{K} \right)^\delta \right] \quad (1.5)$$

Where  $\delta$  is positive exponent and it describes the degree of intraspecific competition.

In this paper, we have considered a predator-prey model, where the prey is vulnerable to strong Allee effect. In this situation, it is quite natural that predator must try to manage some alternative food source. This is obviously not very easy, and therefore, there must be serious intraspecific competition among predators. Also, as there is strong Allee effect in prey, the mutual interference among predators must play a serious role. From this viewpoint, we have considered a growth function of the

form (1.5) and a Beddington- DeAngelis functional response for predators.

The rest of the paper is structured as follows. In section 2, we present a brief sketch of the construction of the model and the biological relevance of it. In section 3, positivity and boundedness of the basic deterministic model is discussed. Some theorems on extinction of the populations are presented in section 4. Section 5 deals with the equilibrium points and their stability analysis. In section 7, computer simulation of a variety of numerical solutions of the system is presented. Section 8 contains the general discussions on the results and biological implications of our mathematical findings.

## II. THE MATHEMATICAL MODEL

Before we introduce the mathematical model, let us describe the basic assumptions that we made to formulate it.

1. The biological system we consider is composed of a single prey population whose density at time  $T$  is denoted by  $X$ . The density of the predator at time  $T$  is denoted by  $Y$ .
2. We consider a multiplicative Allee effect in prey population growth.
3. As the prey is subject to Allee effect, it is assumed that the predator manage some additional food source (for which there is serious intraspecific competition among them), and there is mutual interference among the predators for the prey  $X$  also. Therefore, we assume that the predator follows a modified logistic growth of the form (1.5) and the predator consumes the prey with a Beddington-DeAngelis type functional response.

The above considerations motivate us to introduce the following predator-prey model under the framework of the following set of nonlinear ordinary differential equations:

$$\frac{dX}{dT} = rX \left( 1 - \frac{X}{K} \right) \left( \frac{X}{K_0} - 1 \right) - \frac{\alpha XY}{a_1 + b_1 X + c_1 Y}, X(0) > 0$$

$$\frac{dY}{dT} = sY \left[ 1 - \left( \frac{Y}{L} \right)^\delta \right] + \frac{\alpha \gamma XY}{a_1 + b_1 X + c_1 Y}, Y(0) > 0 \quad (2.1)$$

Here  $r$  is the intrinsic growth rate,  $K$  is the carrying capacity, and  $K_0$  is the Allee threshold for the prey.  $s$  is the intrinsic growth rate of the predator,  $L$  its carrying capacity, and  $\delta$  describes the degree of intraspecific competition. The parameter  $\alpha$  denotes the maximal growth rate of the predator; and  $\gamma$  is the biomass conversion rate of the predator. The parameters  $b_1$  and  $c_1$  respectively describe the effect of handling time and the magnitude of interference among predators; and  $a_1$  is the half saturation constant. We also assume that all parameters are positive.

To reduce the number of parameters, we use the following scaling

$$\frac{X}{K} = x, \frac{Y}{L} = y, t = r \frac{K}{K_0} T.$$

Then the system (2.1) becomes

$$\frac{dx}{dt} = x(1-x)(x-m) - \frac{pxy}{1+bx+cy}, x(0) > 0$$

$$\frac{dy}{dt} = ey(1-y^\delta) + \frac{qxy}{1+bx+cy}, y(0) > 0 \quad (2.2)$$

Where

$$m = \frac{K_0}{K}, p = \frac{\alpha K_0 L}{r K a_1}, b = \frac{b_1 K}{a_1}, c = \frac{c_1 L}{a_1}, e = \frac{K_0}{r K}, q = \frac{\alpha \gamma K_0}{r a_1}, \text{ and } x(0) > 0, y(0) > 0, z(0) > 0.$$

### III. POSITIVITY AND BOUNDEDNESS

Positivity and boundedness of a model guarantee that the model is biologically well behaved. It is easy to notice that the functions on the right side of each of the equations in (2.2) are continuously differentiable in  $R_+^2$ . Therefore the solution of (2.2) with a positive initial condition exists and is unique. For positivity of the system (2.2), we have the following theorem.

**Theorem 3.1** All solutions of the system (2.2) that start in  $R_+^2$  remain positive forever. The proof is simple and therefore it is omitted. Now we prove a useful result.

**Theorem 3.2**  $\limsup_{t \rightarrow \infty} x(t) \leq 1$ .

*Proof. Case-I.* Let  $x(0) \leq 1$ . We claim that  $x(t) \leq 1$  for all  $t \geq 0$ .

If possible, assume that our claim is not true. Then it is possible to find two positive real numbers  $t'$  and  $t''$  such that  $x(t') = 1$  and  $x(t) > 1$  for all  $t \in (t', t'')$ .

Now, for all  $t \in (t', t'')$ , we have from the first equation of (2.2)

$$x(t) = x(0) \exp \left( \int_0^t \phi(x(s), y(s)) ds \right), \text{ where}$$

$$\phi(x(t), y(t)) = (1-x(t))(x(t)-m) - \frac{py(t)}{1+bx(t)+cy(t)}.$$

This implies that

$$x(t) = x(0) \left[ \exp \left( \int_0^{t'} \phi(x(s), y(s)) ds \right) \right] \left[ \exp \left( \int_{t'}^t \phi(x(s), y(s)) ds \right) \right]$$

$$= x(t') \exp \left( \int_{t'}^t \phi(x(s), y(s)) ds \right), \text{ for all } t \in (t', t'')$$

Since  $m < 1$ , we have  $\phi(x(t), y(t)) < 0$  for all  $t \in (t', t'')$ . Consequently, we have  $x(t) < x(t')$ , where  $x(t') = 1$ .

This is contrary to the assumption that  $x(t) > 1$  for all  $t \in (t', t'')$ . Thus our claim is true.

*Case-II.* Let  $x(0) > 1$ . We claim that

$\limsup_{t \rightarrow \infty} x(t) \leq 1$ . If possible, assume that this claim is false. Then  $x(t) > 1$  for all  $t > 0$ . So  $\phi(x(t), y(t)) < 0$

(where  $\phi$  has the same expression as in Case-I); and consequently, we have from the first equation of (2.2) that

$$x(t) = x(0) \exp \left( \int_0^t \phi(x(s), y(s)) ds \right) < x(0)$$

Also from the first equation of (2.2), we obtain

$$\frac{1}{x} \frac{dx}{dt} < (x(0) - m)x(1-x) < 0, \text{ where } x(0) - m > 0.$$

This implies that  $\limsup_{t \rightarrow \infty} x(t) \leq 0$ , which is contradictory to our assumption. Therefore our claim is true.

From the above two cases, we have  $\limsup_{t \rightarrow \infty} x(t) \leq 1$ .

The following theorem ensures the boundedness of the system (2.2).

**Theorem 3.3** All solutions of the system (2.2) that starts in  $R_+^2$  are uniformly bounded.

*Proof.* Let  $W(x, y) = qx + py$ . Then

$$\frac{dW}{dt} \leq q(1+m)x + pey(1-y^\delta)$$

For any  $\mu > 0$ , we have

$$\frac{dW}{dt} + \mu W \leq q(1+m+\mu)x + pey(1+\mu-y^\delta).$$

Now for all  $\delta > 0$ ,  $\max\{y(1+\mu-y^\delta)\} = v$ , where  $v = \delta \left(\frac{1+\mu}{1+\delta}\right)$ .

Therefore,  $\frac{dW}{dt} + \mu W \leq \sigma$ , where  $\sigma = q(1+m+\mu) + pev$ .

Applying the theory of differential inequalities, we obtain

$$0 \leq W(x, y) \leq \frac{\sigma}{\mu} + \frac{W(x(0), y(0))}{e^{\mu t}}$$

$$\text{and for } t \rightarrow \infty \quad 0 \leq W \leq \frac{\sigma}{\mu}.$$

Thus, all the solutions of (2.2) enter into the region

$$B = \left\{ (x, y) : 0 \leq W(x, y) \leq \frac{\sigma}{\mu} + \varepsilon, \text{ for any } \varepsilon > 0 \right\}.$$

Hence the theorem.

#### IV. EXTINCTION SCENARIOS

In this section, we find some conditions for extinction of the prey. Here we use the symbols  $\bar{x}$  and  $\underline{y}$  to represent  $\limsup_{t \rightarrow \infty} x(t)$  and  $\liminf_{t \rightarrow \infty} y(t)$ , respectively. We frequently use the fact that  $x \leq 1$ , which is proved in Theorem 3.2.

It is quite obvious that if, after certain time, the prey population density lies below the Allee threshold (moreover there is attack of predator), then it is really impossible for the prey to survive. This fact is represented in mathematical terms in the following theorem.

**Theorem 4.1** If  $\bar{x} < m$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof:* If possible, let  $\lim_{t \rightarrow \infty} x(t) = \mu > 0$ .

The definition of  $\bar{x}$  implies that for any  $\varepsilon$  satisfying

$$0 < \varepsilon < m - \bar{x} \text{ there exists } t_\varepsilon > 0$$

$$\text{such that } x(t) < \bar{x} + \varepsilon \text{ for } t > t_\varepsilon$$

Then for  $t > t_\varepsilon$  we have from the first equation (2.2) that

$$\frac{dx}{dt} = x(t) \exp \left\{ \int_0^t \left[ (1-x(s))(x(s)-m) - \frac{py(s)}{1+bx(s)+cy(s)} \right] ds \right\}$$

$$< x(0) \exp \left( \int_0^t (\bar{x} + \varepsilon - m) ds \right)$$

$$< x(0) \exp \{ -(m - \bar{x} - \varepsilon)t \} \rightarrow 0 \text{ as } t \rightarrow \infty,$$

which is a contradiction. This proves the theorem.

**Remark.** If the condition of the above theorem is satisfied, then the predator have no vital role in leading the prey to extinction, because the Allee effect is enough to do this (of course, the predator might expedite the process of extinction of the prey). The following theorem shows that the predator might also play a key role to prompt the prey to die out.

**Theorem 4.2** - If  $\underline{y} > \frac{1}{c} \left[ \frac{c(1+2b)}{cm+p-2c} - 1 - b \right] > 0$ ,

then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof:* if possible, let  $\lim_{t \rightarrow \infty} x(t) = \mu > 0$ .

Since  $\bar{x} \leq 1$ , for any  $0 < \varepsilon < 1 - m$ , there

exists  $t_\varepsilon > 0$  such that  $y(t) > y - \varepsilon$

for  $t > t_\varepsilon$ .

Then, for  $t > \max\{t_\varepsilon, t_\varepsilon'\}$ ,

we have from equation of (2.2) that

$$\frac{dx}{dt} < x(1 + \varepsilon - m) - \frac{pxy}{1 + bx + cy}$$

$$< x(1 + \varepsilon - m) - \frac{pxy}{1 + b(1 + \varepsilon) + cy}$$

$$< x \left[ (1 + \varepsilon - m) - \frac{p}{c} \frac{cy}{1 + b(1 + \varepsilon) + cy} \right],$$

$$< x \left[ (2 - m) - \frac{p}{c} \left\{ \frac{1 + b(1 + \varepsilon) + cy - 1 - b(1 + \varepsilon)}{1 + b(1 + \varepsilon) + cy} \right\} \right],$$

### V. EQUILIBRIA AND THEIR STABILITY

In this section, we find the equilibrium points of the system (2.2) and study their stability. The following lemma gives the boundary equilibrium points.

Lemma 5.1 The trivial equilibrium  $E_0(0, 0)$

of the system (2.2) always exists. There are two predator-free equilibrium points  $E_1(1, 0)$  and  $E_2(m, 0)$ , and one prey-free equilibrium point  $E_3(0, 1)$ , each of which also exists unconditionally.

For the existence of the interior or coexistence equilibrium  $E^*(x^*, y^*)$ , we have the following lemma.

Lemma 5.2 If the equation

$$qx\{p - c(1 - x)(x - m)\} - e\{[p - c(1 - x)(x - m)]^\delta - \{(1 - x)(x - m)\}^\delta (1 + bx^\delta)\} \times [(1 + bx)\{p - (1 - x)(x - m)\} + c(1 - x)(x - m)(1 + bx)] = 0 \quad (5.1)$$

has a positive root, then the first component  $x^*$

of the interior equilibrium  $E^*(x^*, y^*)$  exists,

and it is positive root of (5.1). Further

$$\text{if } (1 - x^*)(x^* - m)\{p - c(1 - x^*)(x^* - m)\} > 0,$$

$$\text{then } y^* \text{ exist and is given by } y^* = \frac{(1 - x^*)(x^* - m)(1 + bx^*)}{p - c(1 - x^*)(x^* - m)},$$

The Jacobian matrix  $J(E_0)$  at  $E_0(0, 0)$  is given by

$$J(E_0) = \begin{pmatrix} -m & 0 \\ 0 & e \end{pmatrix}.$$

One eigen value of  $J(E_0)$  is positive and the other is negative. This indicates that  $E_0$  is a saddle point.

The Jacobian matrix  $J(E_1)$  at  $E_1(1, 0)$  is given by

$$J(E_1) = \begin{pmatrix} m - 1 & -\frac{p}{1 + b} \\ 0 & \frac{q}{1 + b} + e \end{pmatrix}.$$

Clearly,  $E_1$  is also a saddle point (as  $m < 1$ ).

The equilibrium point  $E_2(m, 0)$  has the Jacobian matrix

$$J(E_2) = \begin{pmatrix} m(1 - m) & -\frac{p}{1 + bm} \\ 0 & \frac{qm}{1 + bm} + e \end{pmatrix}$$

As  $m < 1$ , both the eigenvalues are positive, and hence  $E_2$  is always unstable. The equilibrium point  $E_3(0, 1)$  has the Jacobian matrix

$$J(E_3) = \begin{pmatrix} -m - \frac{p}{1 + c} & 0 \\ \frac{q}{1 + c} & -e\delta - \frac{qc}{(1 + c)^2} \end{pmatrix}.$$

As both the eigenvalues are negative,  $E_3$

is locally asymptotically stable.

In the following, we give a very

simple criterion for global stability of  $E_3(0, 1)$ .

Theorem 5.3 If  $\bar{x} < m$  then

$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (0, 1).$$

Proof. It is already established in Theorem 4.1 that, if  $x < m$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

From the second equation of (2.2), we have

$$\frac{dy}{dt} = \left[ e(1 - y^\delta) + \frac{qx}{1 + bx + cy} \right] y, \\ \geq ey(1 - y^\delta).$$

This indicates that  $\liminf_{t \rightarrow \infty} y(t) \geq 1$

Since  $\lim_{t \rightarrow \infty} x(t) = 0$ , therefore,

for any  $\varepsilon > 0$ , there exists  $t_\varepsilon > 0$

such that  $x(t) < \varepsilon$

for  $t > t_\varepsilon$ . Then, for  $t > t_\varepsilon$ ,

we have from the second equation of (2.2) that

$$\frac{dy}{dt} = \left[ e(1 - y^\delta) + \frac{qx}{1 + bx + cy} \right] y,$$

$$< ey \left( 1 - y^\delta + \frac{q}{e} \right)$$

$$< ey \left( 1 + \frac{q\varepsilon}{e} - y^\delta \right).$$

This implies that,  $\limsup_{t \rightarrow \infty} y(t) \leq 1 + \frac{q\varepsilon}{e}$

Letting  $\varepsilon \rightarrow 0$ , we get  $\lim_{t \rightarrow \infty} y(t) = 1$ .

Finally, we consider the stability issue of the most important equilibrium  $E^*(x^*, y^*)$ . We have the following Jacobian matrix at  $E^*(x^*, y^*)$ :

$$J(E^*) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

Where

$$a_{11} = x^*(1 + m - 2x^*) + \frac{bpx^*y^*}{(1 + bx^* + cy^*)^2}$$

$$a_{12} = \frac{p(1 + bx^*)x^*}{(1 + bx^* + cy^*)^2},$$

$$a_{21} = \frac{q(1 + cy^*)y^*}{(1 + bx^* + cy^*)^2},$$

$$a_{22} = -e\delta y^{*\delta} - \frac{qcy^{*2}}{(1 + bx^* + cy^*)^2}$$

The characteristic equation of  $J(E^*)$  is

$$\lambda^2 + P\lambda + Q = 0,$$

Where

$$P = -\text{tr}J(E^*) = -a_{11} - a_{22}$$

$$Q = \det J(E^*) = a_{11}a_{22} - a_{12}a_{21}$$

Then we have the following theorem guaranteeing the stability of  $E^*(x^*, y^*)$ .

Theorem 5.4 The necessary and sufficient

condition for local asymptotic stability

of the interior equilibrium  $E^*(x^*, y^*)$  is

that  $P > 0$  and  $Q > 0$ . It would be interesting

if we can establish some sort of global behaviour

of the interior equilibrium. Let  $y > 0$  and

$$\Omega = \{(x, y) \in R^2 : 0 < x < 1, y > \underline{y}\}.$$

Then we have the following theorem.

Theorem 5.5 If  $E^*(x^*, y^*)$  is locally asymptotically

stable with  $e(\delta + 1)y\delta > 2 + m + e + q(1 + b)$ ,

then  $E^*$  attracts all solutions of the system (2.2) lying in  $\Omega$ .

Proof. Let us write first equation of the system (2.2)

$$\text{as } \frac{dx}{dt} = P(x, y), \text{ and the second equation}$$

$$\text{as } \frac{dy}{dt} = Q(x, y). \text{ Then for all } (x, y) \in \Omega, \text{ we notice that}$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x} = -3x^2 + 2(1 + m)x - m - \frac{p(1 + cy)y}{(1 + bx + cy)^2}$$

$$+ e - (1 + \delta)y^\delta + \frac{qx(1 + bx)}{(1 + bx + cy)^2}$$

$$\leq 2(1 + m)x - m + e - e(1 + \delta)y^\delta + qx(1 + bx)$$

$$\leq 2 + m + e - e(1 + \delta)\underline{y}^\delta + q(1 + b) < 0$$

Therefore, by Bendixson's criterion, there is no periodic orbit in  $\Omega$ . Hence the theorem follows from the Poincaré-Bendixson theorem.

### VI. NUMERICAL SIMULATION

In this section, we present computer simulations of some solutions of the system (2.2). These simulations are performed to validate some of the analytical findings of the last two sections.

First, we take the parameters of the system (2.2) as  $m = 0.2, p = 1, b = 1, c = 1, e = 0.2, q = 1, \delta = 2$  and  $(x(0), y(0)) = (0.5, 0.5)$ . Then local asymptotic stability of  $E_3(0, 1)$  is shown in Figure 1.

Next we consider the stability of the interior equilibrium point. For  $m = 0.2, p = 0.1, b = 0.4, c = 0.5, e = 0.2, q = 0.1, \delta = 2$ , the system (2.2) has an interior equilibrium

Point  $E^*(0.9196, 1.1131)$ , which is locally asymptotically stable. The corresponding phase portrait for different choices of  $(x(0), y(0))$  is depicted in Figure 2. Clearly the trajectories converge to  $E^*$ . The figure also shows that the equilibrium point is a stable node. Figure 3 shows the behavior of  $x$  and  $y$  with time, when  $(x(0), y(0)) = (0.6, 1.4)$ , and it is evident that  $(x, y)$  approaches  $(x^*, y^*)$  in finite time.

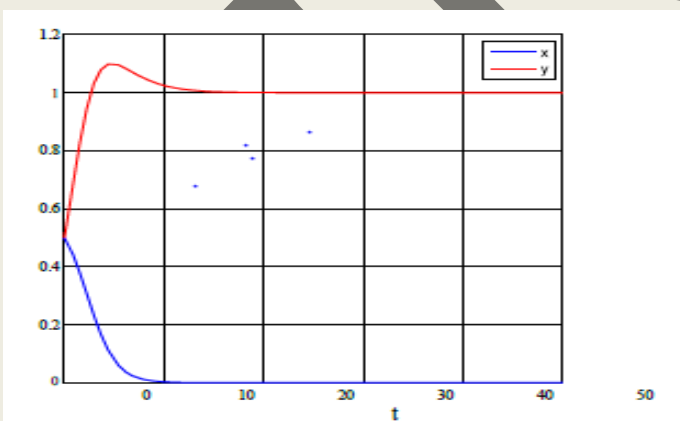


Figure 1: Behaviour of the system (2.2) with time when  $m = 0.2, p = 1, b = 1, c = 1, e = 0.2, q = 1, \delta = 2, x(0) = 0.5$  and  $y(0) = 0.5$ .

### VII. CONCLUDING REMARKS

About an entire century has already been elapsed on understanding and analyzing the basic rule between live food and its eater. Though it is not possible to construct a mathematical model that will fit entirely any natural subsystem, but there always has been a constant endeavour from researchers to find out the most suitable model that might describe and forecast natural phenomena.

It was more than eighty years ago when the concept of Allee effect had drawn the attention of the scientists. Theoretical ecology remained silent for a long time on this issue. Recently, modelers have felt the necessity of modelling of Allee effect. Some models have been developed and cultured in last few years (for details, see [29]). In this paper, we have considered a predator-prey model where the prey is susceptible to Allee effect. The predator has a modified logistic growth and at the same time it consumes the prey following

Beddington-DeAngelis functional response. It is assumed that predators have to fight among themselves for food. The number of parameters of the model has been reduced by suitable scaling. Then the dynamical behaviour of the resulting model (2.2) is studied. Existence and uniqueness of solutions of the model are discussed. It is shown (in Theorem 3.1 and Theorem 3.3) that the solutions of the system (2.2) remains

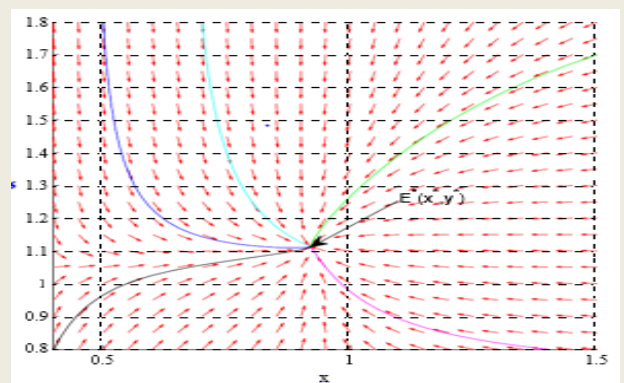


Figure 2: Here  $m = 0.2, p = 0.1, b = 0.4, c = 0.5, e = 0.2, q = 0.1, \delta = 2$ . Phase portrait of the system (2.2) for different choices of  $x(0)$  and  $y(0)$  showing stable behaviour of  $E^*(0.9196, 1.1131)$ .



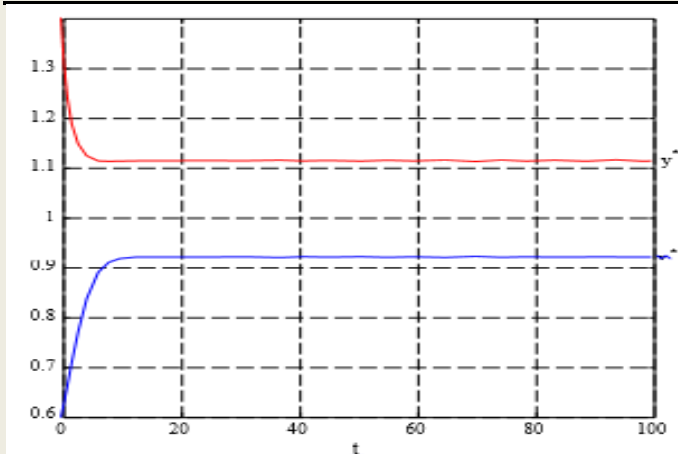


Figure 3: Here the values of the parameters are as in Figure 2. When  $(x(0), y(0)) = (0.6, 1.4)$ , both the populations converge to their equilibrium-state values in finite time. The blue curve represents  $x$  and the red one represents  $y$ .

Non-negative forever, and they are uniformly bounded. These, in turn, imply that the system is biologically well-behaved. We have derived some results on extinction of prey and predator. It is seen that if there is a very strong Allee effect, then it is almost impossible for the prey to survive. Also, an aggressive predator might cause extinction of the prey. If the prey dies out, it is quite natural that the predator will tend towards its carrying capacity (it is also evident from Theorem 5.3).

The stability analysis of the equilibrium points is very important from practical point of view. In this context, the interior equilibrium point is of extreme importance. The importance of ecological balance in nature has now been felt around the entire globe. Stability analysis of the interior equilibrium might provide the conditions for ecological balance in nature. Here we have presented the stability analysis of all the equilibrium points of the system (2.2). We have derived some global results also. The conditions of Theorems 5.4 and 5.5 may be used for ecological balance in nature.

### References

[1] P. Aguirre, E. González-Olivares, and E. Sáez, *Two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect*, Nonlinear Anal. RWA, 10 (2009a), 1401-1416.

[2] P. Aguirre, E. González-Olivares, and E. Sáez, *Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect*, SIAM J. Appl. Math., 69 (2009b), 1244-1262.

[3] W. Allee, *Animal aggregations*, Quart. Rev. Biol., 2 (1927a), 367-398.

[4] W. Allee, *Studies in animal aggregations: some physiological effects of aggregation on the brittle starfish, Ophioderma brevispina*, J. Exp. Zool., 48 (1927b), 475-495.

[5] W. Allee, *Studies in animal aggregations: mass protection against colloidal silver among goldfishes*, J. Exp. Zool., 61 (1932), 185-207.

[6] A.D. Bazykin, F.S. Berezovskaya, A.S. Isaev, and R.G. Khlebopros, *Dynamics of forest insect density: bifurcation approach*, J. Theor. Biol., 186 (1997), 267-278.

[7] J.R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, J. Animal Ecol., 44 (1975), 331-340.

[8] S.P. Bera, A. Maiti, and G.P. Samanta, *Dynamics of a food chain model with herd behaviour of the prey*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences (to appear)

[9] L. Berec, E. Angulo, and F. Courchamp, *Multiple allee effects and population management*, Trends Ecol. Evol., 22 (2006), 185-191.

[10] C.W. Clark, *Mathematical Bioeconomic: The Optimal Management of Renewable Resources*, Wiley, New York, 1990.

[11] C.W. Clark, *The Worldwide Crisis in Fisheries: Economic Models and Human Behavior*, Cambridge University Press, Cambridge, 2007.

[12] T.H. Clutton-Brock, D. Gaynor, G.M. McIlrath, A.D.C. Maccoll, R. Kansky, P. Chadwick, M. Manser, J.D. Skinner, and P.N.M. Brotherton, *Predation, group size and mortality in a cooperative mongoose, Suricata suricatta*, J. Anim. Ecol., 68 (1999), 672-683.

- [13] E.D. Conway, and J.A. Smoller, *Global analysis of a system of predator-prey equations*, SIAM J. Appl. Math., 46 (1986), 630-642.
- [14] F. Courchamp, T. Clutton-Brock, and B. Grenfell, *Inverse dependence and the Allee effect*, Trends Ecol. Evol., 14 (1999), 405-410.
- [15] F. Courchamp, L. Berec, and J. Gascoigne, *Allee Effects in Ecology and Conservation*, Oxford University Press, Oxford, 2008.
- [16] D.L. DeAngelis, R.A. Goldstein, R.V. O'Neill, *A model for trophic interaction*, Ecology, 56 (1975), 881-892.
- [17] C.S. Elton, *Animal Ecology*, Sidgwick and Jackson, London, 1927.
- [18] J.D. Flores, J. Mena-Lorca, B. González-Yañez, and E. González-Olivares, *Consequences of depensation in a Smith's bioeconomic model for open-access fishery*, In R. Mondaini and R. Dilao (Eds.), Proceedings of international symposium on mathematical and computational biology (pp. 219-232), E-papers Servios Editoriais Ltda, 2007.
- [19] E. González-Olivares, and A. Rojas-Palma, *Multiple Limit Cycles in a Gause type predator-prey model with Holling type III functional response and Allee effect on prey*, Bull. Math. Biol., 73 (2011), 1378-1397.
- [20] C. S. Holling, *The components of predation as revealed by a study of small mammal predation of the European pine sawfly*, Can. Entomol., 91 (1959a), 293-320.
- [21] C. S. Holling, *Some characteristics of simple types of predation and parasitism*, Can. Entomol., 91 (1959b), 385-398.
- [22] M. Kot, *Elements of Mathematical Biology*, Cambridge University Press, Cambridge, 2001.
- [23] M. Liermann, and R. Hilborn, *Depensation: evidence, models and implications*, Fish Fish., 2 (2001), 33-58.
- [24] R.L. Lindeman, *The trophic-dynamic aspect of ecology*, Ecology 23 (1942), 399-417. [25] A. Lotka, *Elements of Physical Biology*, Williams and Wilkins, Baltimore, 1925.
- [26] N. Macdonald, *Biological Delay Systems: Linear Stability Theory*, Cambridge University Press, Cambridge, 1989.
- [27] A. Maiti, and G.P. Samanta, *Deterministic and stochastic analysis of a prey-dependent predator-prey system*, Internat. J. Math. Ed. Sci. Tech., 36 (2005), 65-83.
- [28] A. Maiti, A. K. Pal, and G. P. Samanta, *Effect of time-delay on a food chain model*, Appl. Math. Comput., 200 (2008), 189-203.
- [29] A. Maiti, P. Sen, D. Manna, and G.P. Samanta, *A Predator-prey system with herd behaviour and strong Allee effect*, Nonlinear Dynamics and Systems Theory (to appear).
- [30] M.S. Mooring, T.A. Fitzpatrick, T.T. Nishihira, and D.D. Reising, *Vigilance, predation risk, and the Allee effect in desert bighorn sheep*, J. Wildlife Management, 68 (2004), 519-532.
- [31] J. D. Murray, *Mathematical Biology*, Springer-Verlag, New York, 1993. [32] E. Odum, *Fundamentals of Ecology*, Saunders, Philadelphia, 1953.
- [33] R. Pearl, and L. Reed, *On the rate of growth of the population of the United States*, Proc. Nat. Acad. Sci., 6 (1920), 275-288.
- [34] P.W. Prince, C.E. Bouton, P. Gross, B.A. McPherson, J.N. Thompson, and A.E. Weis, *Interactions among three trophic levels: influence of plants on interactions between insect herbivores and natural enemies*, Annu. Rev. Ecol. Syst., 11 (1980), 41-65.
- [35] F.J. Richards, *A Flexible Growth Function for Empirical Use*, Journal of Experimental Botany, 10 (1959), 290-300.
- [36] D.J. Rinella, M.S. Wipfli, C.A. Stricker, R.A. Heintz, and M.J. Rinella, *Pacific salmon (*Oncorhynchus* sp.) runs and consumer fitness: growth and energy storage in stream-dwelling salmonids increase with salmon spawner density*, Canad. J. Fish. Aquat. Sci., 69 (2012), 73-84.

- [37] S. Ruan, and D. Xiao, *Global analysis in a predator-prey system with nonmonotonic functional response*, SIAM J. Appl. Math., 61 (2001), 1445-1472.
- [38] P.A. Stephens, and W.J. Sutherland, *Consequences of the Allee effect for behaviour, ecology and conservation* Trends Ecol. Evol., 14 (1999), 401-405.
- [39] P.A. Stephens, W.J. Sutherland, and R.P. Freckleton, *What is the Allee effect?*, Oikos, 87 (1999), 185-190.
- [40] G.A.K. van Voorn, L. Hemerik, M.P. Boer, and B.W. Kooi, *Heteroclinic orbits indicate overexploitation in predator-prey systems with a strong Allee effect*, Math. Biosci., 209 (2007), 451-469.
- [41] P.F. Verhulst, *Notice sur la loi que la population poursuit dans son accroissement*, Correspondance mathématique et physique, 10 (1838), 113-121.
- [42] V. Volterra, *Variazioni e fluttuazioni del numero di individui in specie animali con- viventi*, Mem. Accd. Linc., 2 (1926), 31-113.
- [43] G. Wang, X.G. Liang, and F.Z. Wang, *The competitive dynamics of populations subject to an Allee effect*, Ecol. Model., 124 (1999), 183-192.
- [44] M.H. Wang, and M. Kot, *Speeds of invasion in a model with strong or weak Allee effects*, Math. Biosci., 171 (2001), 83-97.
- [45] J. Wang, J. Shi, and J. Wei, *Predator-prey system with strong Allee effect in prey*, J. Math. Biol., 62 (2011), 291-331.