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SOLVING SOME SIMPLE BUT USEFUL TYPES OF DIFFERENTIAL EQUATION. RICCATI EQUATIONS

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Annotation: This article includes some methods and techniques of solving differential equations, which is effectively used in solving problems of physics and mechanics.
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Introduction

Newton's law of cooling and the equation that we just wrote down is an example of a differential equation. Ideally we would like to solve this equation, namely, find the function f(t) that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equation.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appear. Typically, a scientific theory will produce a differential equation that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

A differential equation is an equation that involves derivatives or differentials. If the equation involves only derivatives of a function of one variable, then it is said to be ordinary. An equation that involves x, y, $y', y'', ...y^{(n)}$, for a function **y** of **x** with nth derivative $\mathbf{y^{(n)}}$ of **y** with respect to **x**, is an ordinary differential equation of order n [1].

First order linear differential equations

A first order linear differential equation has the following form:

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \qquad (1)$$

Wee will find the general solution in form

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$$y = u(x)\cdot v(x) \qquad (2)$$

Hence $u(x)$ and $v(x)$ are arbitrary functions of x. They are while unknown functions.

From (2) we get $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (3)$$

After substitution of (2) and (3) for (1)

$$u\frac{dv}{dx} + v\frac{du}{dx} + u\cdot v\cdot P = Q(x) \Rightarrow \left(\frac{dv}{dx} + P\cdot v\right)u + v\frac{du}{dx} = Q(x) \qquad (4)$$

If $\frac{dv}{dx} + Pv = 0$ then $\frac{dv}{dx} = -Pv \Rightarrow \frac{dv}{v} = -Pdx \Rightarrow$

$$\Rightarrow \ln v = -\int Pdx \Rightarrow \left| \begin{array}{l} \ln v - \ln C = -\int Pdx \\ v = C\cdot e^{-\int Pdx} \end{array} \right| \Rightarrow C = 1 \quad \Rightarrow \quad v = e^{-\int Pdx}$$

Substituting into the equation (4) gives

$$v\frac{du}{dx} = Q(x) \Rightarrow du = \frac{Q(x)}{v(x)}dx \Rightarrow u = \int\frac{Q(x)}{v(x)}dx + C$$

Hence, the **y** is given by the formula $y = u(x)\cdot v(x)$

The general solution is $$y = \left[\int\frac{Q(x)}{v(x)}dx + C\right]v(x)$$

where $$v = e^{-\int P(x)dx}$$

called the **integrating factor**. If an initial condition is given, use it to find the constant C.

Here are some practical steps to follow:

1. If the differential equation is given as

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

rewrite it in the form

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$$\frac{dy}{dx}+p(x)y=q(x)$$

where $p(x)=\frac{b(x)}{a(x)}$ and $q(x)=\frac{c(x)}{a(x)}$

2. Find the integrating factor

$$v=e^{-\int P(x)dx}$$

3. Evaluate the integral $u=\int\frac{Q(x)}{v(x)}dx+C$

Write down the general solution

$$y=\left[\int\frac{Q(x)}{v(x)}dx+C\right]v(x)=\left[\int\frac{Q(x)}{v(x)}dx+C\right]e^{-\int P(x)dx}$$

5. Use the initial condition to find the constant C.

Example 1. Find the particular solution of:

$$\frac{dy}{dx}+\tan(x)y=\cos^2(x); \quad y(0)=2$$

Solution: Let us use the steps:

Step 1: There is no need for rewriting the differential equation. We have

$$p(x)=\tan(x) \quad \text{and} \quad q(x)=\cos^2(x)$$

Step 2: Integrating factor

$$u(x)=e^{\int\tan(x)dx}=e^{-\ln(\cos(x))}=e^{\ln(\sec(x))}=\sec(x)$$

Step 3: We have

$$\int\sec(x)\cos^2(x)dx=\int\cos(x)dx=\sin(x)$$

Step 4: The general solution is given by

$$y=\frac{\sin(x)+C}{\sec(x)}=(\sin(x)+C)\cos(x)$$

Step 5: In order to find the particular solution, we use the initial condition to find C. Indeed, we have

$$y(0)=C=2$$

Therefore the solution is

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$$y = (\sin(x) + 2)\cos(x)$$

Note that you may not have to do the last step if you are asked to find the general solution.

Bernoulli equations

A differential equation of Bernoulli type is written as

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

This type of equation is solved via a substitution. Indeed, let $z = y^{1-n}$

Then easy calculations give

$$z' = (1-n)y^{-n}y'$$

which implies $$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

This is a linear equation satisfied by the new variable **z**. Once it is solved, we will obtain the function $y = z^{\frac{1}{(1-n)}}$. Note that if n > 1, then we have to add the solution y=0 to the solutions found via the technique described above [2].

Let us summarize the steps to follow:

1. Recognize that the differential equation is a Bernoulli equation. Then find the parameter n from the equation;

2. Write out the substitution $z = y^{1-n}$

3. Through easy differentiation, find the new equation satisfied by the new variable **z** :

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$$

4. Solve the new linear equation to find **z**;

5. Go back to the old function **y** through the substitution $y = z^{\frac{1}{(1-n)}}$;

6. If n > 1, add the solution y=0 to the ones we obtained in (4).

7. Use the initial condition to find the particular solution.

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Riccati equations

Before we give the formal definition of **Riccati equations**, a little introduction may be helpful. Indeed, consider the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

If we approximate f(x,y), while x is kept constant, we will get

$$f(x, y) = P(x) + Q(x)y + R(x)y^2 + \cdots\cdots$$

If we stop at **y**, we will get a linear equation. Riccati looked at the approximation to the second degree: he considered equations of the type

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

These equations bear his name, **Riccati equations**. They are nonlinear and do not fall under the category of any of the classical equations. In order to solve a Riccati equation, one will need a particular solution. Without knowing at least one solution, there is absolutely no chance to find any solutions to such an equation. Indeed, let y_1 be a particular solution of

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

Consider the new function z defined by

$$z = \frac{1}{y - y_1}$$

Then easy calculations give

$$\frac{dz}{dx} = -(Q(x) + 2y_1R(x))z - R(x)$$

which is a linear equation satisfied by the new function **z**. Once it is solved, we go back to **y** via the relation

$$y = y_1 + \frac{1}{z}$$

Keep in mind that it may be harder to remember the above equation satisfied by **z**. Instead, try to do the calculations whenever you can.

Example 2. Solve the equation

$$\frac{dy}{dx} = -2 - y + y^2$$

knowing that $y_1 = 2$ is a particular solution.

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Solution: We recognize a Riccati equation. First of all we need to make sure that y_1 is indeed a solution. Otherwise, our calculations will be fruitless. In this particular case, it is quite easy to check that $y_1 = 2$ is a solution. Let

$$z = \frac{1}{y-2}$$

Then we have

$$y = 2 + \frac{1}{z}$$

which implies

$$y' = -\frac{z'}{z^2}$$

Hence, from the equation satisfied by **y**, we get

$$-\frac{z'}{z^2} = -2 - \left(2 + \frac{1}{z}\right) + \left(2 + \frac{1}{z}\right)^2$$

Easy algebraic manipulations give

$$-\frac{z'}{z^2} = \frac{3}{z} + \frac{1}{z^2}$$

Hence

$$z' = -3z - 1$$

This is a linear equation. The general solution is given by

$$z = \frac{\frac{-1}{3}e^{3x} + C}{e^{3x}} = -\frac{1}{3} + Ce^{-3x}$$

Therefore, we have

$$y = 2 + \frac{1}{-\frac{1}{3} + Ce^{-3x}}$$

Note: If one remembers the equation satisfied by **z**, then the solutions may be found a bit faster. Indeed in this example, we have P(x) = -2, Q(x) = -1 and R(x) = 1. Hence the linear equation satisfied by the new function **z** , is

$$\frac{dz}{dx} = -(Q(x) + 2y_1R(x))z - R(x) = -(-1+4)z - 1 = -3z - 1$$

The general first order equation is rather too general, that is, we can't describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of

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the form $y' = \varphi(t, y)$ where φ is a function of the two variables t and y. Under reasonable conditions on φ, such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly [3].

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