EXISTENCE RESULT FOR INITIAL VALUE PROBLEM WITH NONLINEAR FUNCTIONAL RANDOM FRACTIONAL DIFFERENTIAL EQUATION

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ABSTRACT:- In this paper we prove the existence result for initial value problems with nonlinear functional random fractional differential equations under Caratheodory condition .

KEY WORDS: **-** Random fractional differential equation, fractional integral, Caputo fractional derivative etc.
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1. **INTRODUCTION:-**

The linear as well as nonlinear initial value problem of random differential equations have been studied in the literature by the authors since long time refer a Dhage[4-6]. Similarly the fraction differential equation are frequently use in many branches of engineering and science It has been mentioned first by Liouville in a paper from 1832. There are real world phenomena with anomalous dynamics such as signals transmission, network traffic and
so on. In this case the theory of fractional differential equation is a good tool for modeling such as phenomena. For some fundamental result in the theory of fractional differential equations. We refer paper of Lakshmikantham [9, 10, 11] and [13,14].

Let $\mathbb{R}$ denote the real line and Let $I_0 = [-r, 0]$ and $I = [0, T]$ be two closed and bounded interval in $\mathbb{R}$ for some r> 0 and T > 0. Let $J = I_0 \ \ UI$. Let $C(I_0, \mathbb{R})$ denote the space of continuous $\mathbb{R}$ valued function I_0 . We equip the space $C = C(I_0, \mathbb{R})$ with a supremum norm $\|.\|_c$ defined by

$$\|x\|_c = \sup_{t\epsilon I_0} |x(t)|$$

Clearly C is a Banach Space which is also a Banach Space with respect to this norm. For a given $t \epsilon I$ define a continuous R-valued function.

$x_t: I_0 \rightarrow \mathbb{R}$ by $x_t(\theta) = (t + \theta)$, $\theta \epsilon I_0$

Let (Ω, A) be a measurable space i.e. a set Ω with a σ-algebra of subset of Ω and for given a measurable function $x\colon \ \Omega \rightarrow C(J, \mathbb{R})$.

Consider nonlinear functional random fractional differential equations of the form (in short RFDE)

$$\left.\begin{array}{l} {}^{c}D^{\alpha} x(t , \omega)= f(t, x_t(\omega), \omega) \ \text{ a.e } t\in J , \ 0<\alpha<1 \\ x(0, \omega) = x_0(\omega) \end{array}\right\} \quad (1.1)$$

Where x is a random function; x_0 is random, $D^{\alpha}x$ is the Caputo fractional derivative of x with respect to the variable $t \in J$ and $f: J \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is given function.

2. **EXISTENCE RESULT:-**

Let E denote a Banach space with the norm $||.||$ and let Q: E⟶E. We further assume that the Banach space E is separable i.e. E has countable dense subset and let β_E be the $\sigma -$ algebra of Borel subset of E. We say a mapping x: Ω⟶ E is measurable if for any $B \in \beta_E$ one has

$$x^{-1}(B) = \{(\omega, x) \in \Omega \times E : x(\omega, x) \in B\} \in \mathcal{A} \times \beta_E$$

Where $\mathcal{A} \times \beta_E$ is the direct product of the σ- algebras A and β_E those defined in Ω and E respectively.

Let Q: Ω × E⟶ E be a mapping. Then Q is called a random operator if Q (ω, x) is measurable in ω for all x∈E and it is expressed as Q(ω) x = Q(ω, x). A random operator Q (ω) on E is called continuous (resp. compact, totally bounded and completely continuous) If Q(ω, x) is continuous (resp. compact, totally bounded and completely continuous) in x for all ω ϵ Ω.

Lemma 2.1[12]: Let $B_R(0)$ and $\overline{B_R}(0)$ be the open and closed ball centered at origin of radius R in the separable Banach space E and let Q: $\Omega\times \overline{B_R}(0)\longrightarrow$ E be a compact and continuous random operator. Further suppose that there does not exists an uϵ E with ||u||=R such that Q (ω) u = λ u for all λ ϵ Ω where λ > 1. Then the random equation Q(ω)x = x has a random solution, i.e. there is a measurable function $\xi:\Omega\longrightarrow\overline{B_R}(0)$ such that $Q(\omega)\xi(\omega)=\xi(\omega)$ for all ωϵ Ω .

Lemma 2.2[12]: (Carathéodory) Let Q: Ω **×** E ⟶ E be a mapping such that Q(.,x) is measurable for all xϵ E and Q(ω,.) is continuous for all ω ϵΩ Then the map (ω, x) ⟶Q(ω, x) is jointly measurable.

We seek random solution of (1.1) in Banach space C (J, ℝ) of continuous real valued function defined on J. We equip the space C (J, ℝ) with the supremum norm**||.||** defined by

$$||x|| = \sup_{t\epsilon\mathcal{J}}|x(t)|$$

It is known that the Banach space C ($\mathcal{J}$, ℝ) is separable. By L^1 ($\mathcal{J}$, ℝ) we denote the space of Lebesgue measurable real-valued function defined on $\mathcal{J}$. By $||.||_{L^1}$ we denote the usual norm in L^1 ($\mathcal{J}$, ℝ) defined by

$$||x||_{L^1}= \int_0^1 |x(t)|dt.$$

We need the following definition in the sequel.

Definition 2.1: A Carathéodory function $f : \mathcal{J} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ is called random L^1- Carathéodory if for each real number r>0 there is a measurable and bounded function $h_r : \Omega \longrightarrow L^1(\mathcal{J},\mathbb{R})$ such that

$$|f(t, x, \omega)| \leq h_r(t, \omega) \qquad \text{a. e.} t \in \mathcal{J} .$$

Where $|x|\leq r$ and for all ω ϵ Ω

We consider the following set of hypothesis

H1) The function (t, x) → f(t, x, ω) is continuous for a.e. ω∈Ω.

H2) The function $\omega \rightarrow f(t, x, \omega)$ is measurable for a. e. $\omega \in \Omega$.

H3) There exist $y \in \mathbb{R}$ s.t. $x_0(\omega) \in \overline{B_R}(0)$ for a. e. $\omega \in \Omega$ where
$\overline{B_R}(0) = \{ x \in \mathbb{R} : \|x - x_0\| \leq \varepsilon \}$.

H4) There exist K>0 and $x_0 \in \Omega$ s.t.$\|f(t, x, \omega)\| \leq \frac{K}{\Gamma\alpha}$ a.e. $\omega \in \Omega$. And Γ is a gamma function

Our main existence result is

Theorem 2.1: Assume that the hypothesis $H_1 - H_4$ hold. Suppose that there exist a real number R>0 such that

$$R > r_1 \|\gamma(\omega)\|_{L^1} \psi(\mathbb{R}) \quad \ldots (2.1)$$

for all $\omega \in \Omega$ where $r_1 = \max_{t\in[0,1]} r(t)$, $r(t)$ is in the greens function

Then the (1.1) has a random solution defined on $\mathcal{J}$

Proof: - Set E=C ($\mathcal{J}$, $\mathbb{R}$) and define a mapping Q: $\Omega \times E \longrightarrow E$ by

$$Q(\omega)x(t) = x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, x_s(\omega), \omega) ds \quad \ldots (2.2)$$

a.e. $\omega \in \Omega$ and for all $t \in J$. (Equation (2.2) is an Integral representation of (1.1)) Then the solution of (1.1) is fixed point of operator Q .

Define a closed ball $\overline{B_R}$ (0) in E centered at origin with radius R where the real number R satisfies the inequality (2.1). We show that Q satisfies all the condition of lemma 2.1 on $\overline{B_R}$ (0).

First we show that Q is random operator in $\overline{B_R}$ (0). Since $f(t, x_t(\omega), \omega)$ is random Caratheodory and x(t, ω) is measurable, the map $\omega \rightarrow f(t, x_t(\omega), \omega)$ is measurable. Similarly the product $g_\alpha(t-s) f(s, x_s(\omega), \omega)$ of continuous and measurable function is again measurable. Further the integral is a limit of finite sum of measurable function . Therefore the map

$$\Omega \rightarrow x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, x_s(\omega), \omega) ds = Q(\omega)x(t)$$

is measurable.

As a result Q is random operator on $\Omega \times \overline{B_R}$ (0) in to E.

Next we show that the random operator Q(ω) is continuous on $\overline{B_R}$ (0). Let x_n be a sequence of point in $\overline{B_R}$ (0) converging to the point x in $\overline{B_R}$ (0). Then it is sufficient to prove that

$$\lim_{n\to\infty} Q(\omega)\, x_n(t) = Q(\omega)x(t) \quad \text{for all } t \in J,\ \omega \in \Omega.$$

By the dominated convergent theorem we obtain

$$\lim_{n\to\infty} Q(\omega)\, x_n(t) = \lim_{n\to\infty} \left[x_0(\omega) + \int_0^t g_\alpha(t-s)\, f(s, x_n(\omega), \omega)\, ds \right]$$
$$= x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, x_s(\omega), \omega) ds$$
$$= Q(\omega)x(t)$$

For all $t \in J$, $\omega \in \Omega$. This shows that Q(ω) is a continuous random operator on $\overline{B_R}$ (0).

Now we show that Q(ω) is a compact random operator on $\overline{B_R}$ (0).

To finish it, we should prove that $Q(\omega)(\overline{B_R}\ (0))$ is a uniformly bounded equi continuous set in E for each $\omega \in \Omega$. Since the map $\omega \rightarrow \frac{Ka^\alpha}{\Gamma\alpha+1} \leq \frac{\varepsilon}{2}$.

Let $\omega \in \Omega$ be fixed then for any x: $\Omega \rightarrow \overline{B_R}$ (0) has

$$|Q(\omega)x_n(t) - x_o(\omega)| \leq \int_0^t g_\alpha(t-s)\, |f(s, x_n(\omega), \omega)|\, ds$$
$$\leq \frac{K}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1}\, ds$$
$$\leq \frac{Ka^\alpha}{\Gamma\alpha+1} \leq \frac{\varepsilon}{2}$$

Next we show that $Q(\omega)(\overline{B_R}\ (0))$ is equicontinuous set in E for any $x \in \overline{B_R}$ (0), $t_1, t_2 \in J$, $\varepsilon > 0$ we have

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \leq \frac{K}{\Gamma\alpha} \int_0^{t_2} (t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1} d\tau + \frac{K}{\Gamma\alpha} \int_{t_2}^{t_1} (t_1-\tau)^{\alpha-1} d\tau$$
$$\leq \frac{2K}{\Gamma\alpha+1} (t-s)^\alpha < \varepsilon$$

Hence for all $t_1, t_2 \in J$

$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \rightarrow 0$ as $t_1 \rightarrow t_2$ uniformly for all $x \in \overline{B_R}$ (0).

Therefore $Q(\omega)\overline{B_R}$ (0) is . Then we know it is compact by Arzela – Ascoli theorem for each $\omega \in \Omega$. Consequently Q(ω) is a completely continuous random operator on $\overline{B_R}$ (0).

Finally we suppose there exist such an element u in E with $\|u\| = R$ satisfying $Q(\omega)u(t) = \lambda\, u(t, \omega)$ for some $\omega \in \Omega$ and $\lambda > 1$. Now for this $\omega \in$ we have

$$\lambda\, u(t, \omega) = Q(\omega)u(t)$$
$$|u(t, \omega)| \leq \frac{1}{\lambda} |Q(\omega)u(t)|$$
$$\leq \frac{1}{\lambda} |x_0(\omega) + \int_0^t g_\alpha(t-s) f(s, x_s(\omega), \omega) ds|$$
$$\leq \frac{1}{\lambda} x_0(\omega) + \frac{1}{\lambda} \int_0^t g_\alpha(t-s) f(s, x_s(\omega), \omega) ds$$
$$\leq \frac{1}{\lambda} x_0(\omega) + \frac{ka^\alpha}{\lambda\Gamma\alpha+1}$$
$$\leq \rho$$

For all $t \in J$ where $\rho = \frac{1}{\lambda} x_0(\omega) + \frac{ka^\alpha}{\lambda\Gamma\alpha+1}$

This contradicts to inequality (2.1) this all the condition of lemma 2.1 are satisfied.

Hence random equation Q(ω) x(t) = x(t,ω) has a random solution in $\overline{B_R}$ (0) i.e. there exist a measurable function ξ : $\Omega \rightarrow \overline{B_R}$ (0) such that $Q(\omega)\xi(t) = \xi(t)$ for all $\omega \in \Omega$ and $t \in J$. As a result RFDE (1.1) has a random solution defined on J. This completes the proof.

REFERENCES

[1] T.T.Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.

[2] R. P. AGARWAL, M BENCHOHRA, S. HAMANI, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math. DOI 10.1007/s10440-008-9356-6.

[3] R. A. KHAN, Existence and approximation of solutions of nonlinear problems with integral boundary conditions, Dynamic Systems and Applications, **14** (2005), 281–296.

[4] B.C. Dhage, S.K. Ntouyas, Existence and attractivity results for nonlinear first order random differential equations, Opuscula Math. 30 (2010) 4, 411–429.

[5] B.C. Dhage, On global existence and attractivity results for nonlinear random integral equations, Panamer. Math. J. 19 (2009) 1, 97–111.

[6] G.S. Ladde, V. Lakshmikantham, Random Differential Inequalities, Academic Press, New York, 1980.

[7] B.C. Dhage, On n-th order nonlinear ordinary random differential equations, Nonlinear Oscil. 13 (2011) 4, 535–549.

[8] V. Lupulescu, S.K. Ntoukas, Random fractional differential equations, International Electronic Journal of Pure and Applied Mathematics, 4 (2012) 2, 119–136.

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006
[10] V. Lakshmikantham, A.S. Vatsala, Theory of fractional differential inequalities and applications, Commun. Appl. Anal. 11 (2007), 395–402.
[11] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008), 2677–2682.
[12] B.C.Dhage, S.V. badgire, S.K.Ntouyas, Periodic boundary valu problem of second order random differential equation .Electron. J. Qaul. Theory Diff.Equ.,21(2009), 1-14.
[13] M.K.Bhosale , R.N.Ingle,Second Order Nonlinear Functional Random Differential Equation, Journal of Global research in Mathematical Archives, 9(2013), 15-22 .
[14] M.K.Bhosale , R.N.Ingle,Existence Theory For Second Order Nonlinear Functional Random Differential Equation In Banach Algebra, IOSR-JM, 11(2015) 6-12.