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CERTAIN CLASSES OF UNIVALENT FUNCTIONS DEFINED BY RUSCHEWEYH OPERATOR WITH SOME MISSING COEFFICIENT

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ABSTRACT:
In this paper, we introduce the class $O_n(\alpha,\beta,\gamma,\xi)$ by using the Ruscheweyh Operator. The aim of this paper is to study some properties of this class,like, coefficient inequality, Hadamard products, Extreme points, radius of starlikeness and convexity, closure theorem of a class of analytic and univalent function in the unit disc. The results obtained here are found to be sharp.
KEY WORDS: Analytic functions, Univalent functions, Hadamard product, Starlike functions, Rescheweyh operator.

1. INTRODUCTIONS:

Let S denote the class of function

$f(z)=z+\sum_{k=2}^{\infty}a_{2k}z^{2k}$ which are analytic and univalent in the unit disc

$U=\{z:|z|<1\}$. For $g(z)=z+\sum_{k=2}^{\infty}b_{2k}z^{2k}$ the convolution or Hadamard product of $f(z)$ and $g(z)$ is defined by $(f*g)(z)=z+\sum_{k=2}^{\infty}a_{2k}b_{2k}z^{2k}, z\in U$.

Let $D^n f(z)$ denote the n^{th} orderderivative introduced by Ruscheweyh [6].
The Ruscheweyh derivative is defined as follows: $D^n : S\to S$ such that

$$D^n f(z)=\frac{z}{(1-z)^{n+1}} * f(z),\quad n>-1$$

$$=\frac{z(z^{n-1}f(z))^n}{n!}\ ,\ n\in N_0=0,1,2\ldots$$

$$=z+\sum_{k=2}^{\infty}a_{2k}\partial(n,k)z^{2k}\ \text{, Where}$$

$$\partial(n,k)=\binom{n+k-1}{n}.$$

Notice that, $\frac{z}{(1-z)^{n+1}}=z+\sum_{k=2}^{\infty}\partial(n,k)z^{2k}$, where $n>-1$ and $D^0 f(z)=f(z), D'f(z)=zf'(z)$.

We aim to study the class $O_n(\alpha,\beta,\gamma,\xi)$ which consists of functions $f\in S$ and satisfy

$$\left|\frac{z((D^n f(z))'-1)}{z\xi((D^n f(z))'-1)+\{z(D^n f(z))'-D^n f(z)\}+(\alpha-\beta)}\right|<\gamma,\qquad z\in U$$

for

$0\le\xi<1, 0<\alpha\le 1, 0\le\beta<1, 0<\gamma<1, n\in N_0$.

The investigation here is motivated by M.Darus [1].
Next we characterize the class $O_n(\alpha,\beta,\gamma,\xi)$ by proving the Coefficient Inequality.

1. COEFFICIENT INEQUALITY:

Theorem 1:
Let $f\in S$.Then $f\in O_n(\alpha,\beta,\gamma,\xi)$ if and only if

$$\sum_{k=2}^{\infty}\left[2k(1-\gamma\xi-\gamma)+\gamma\right]\partial(n,k)a_{2k}\le\gamma(\alpha-\beta)\qquad(1.1)$$

for $0\le\xi<1, 0<\alpha\le 1, 0\le\beta<1, 0<\gamma<1, n\in N_0$

$$\partial(n,k)=\binom{n+k-1}{n}$$

The result (1.1) is sharp for the function

$$f(z)=z+\frac{\gamma(\alpha-\beta)}{\left[2k(1-\gamma\xi-\gamma)+\gamma\right]\partial(n,k)}z^{2k}\ ,\ k\ge 2$$

Proof: Assume inequality (1.1) is true then

$$\left|z(D^n f(z))'-1)\right| - \gamma \left|\begin{matrix} z\xi(D^n f(z))'-1)+ \\ \{z(D^n f(z))' - D^n f(z)\} \\ +(\alpha-\beta) \end{matrix}\right|$$

$$\le \left|\sum_{k=2}^{\infty} 2k\,\partial(n,k) a_{2k} z^{2k}\right| -$$

$$\gamma \left|\begin{matrix} \xi\sum_{k=2}^{\infty} 2k\,\partial(n,k) a_{2k} z^{2k} \\ +\sum_{k=2}^{\infty} 2k\,\partial(n,k) a_{2k} z^{2k} \\ -\sum_{k=2}^{\infty} a_{2k}\,\partial(n,k)\, z^{2k} + (\alpha-\beta) \end{matrix}\right|$$

$$\le \sum_{k=2}^{\infty} [2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k) a_{2k} - \gamma(\alpha-\beta) \le 0$$

by maximum modulus principle,

$\therefore\ f \in O_n(\alpha,\beta,\gamma,\xi)$.

Conversely, Let $f \in O_n(\alpha,\beta,\gamma,\xi)$.Then

$$\left|\frac{z(D^n f(z))'-1)}{z\xi(D^n f(z))'-1) + \{z(D^n f(z))' - D^n f(z)\} + (\alpha-\beta)}\right| < \gamma, \quad z \in U$$

that is

$$\frac{\left|\sum_{k=2}^{\infty} 2k\,\partial(n,k)\, a_{2k} z^{2k}\right|}{\left|\sum_{k=2}^{\infty} (2k\xi+2k-1)\partial(n,k)\, a_{2k} z^{2k} + (\alpha-\beta)\right|} < \gamma \tag{1.2}$$

Using the fact that $\left|\mathrm{Re}(f(z))\right| \le \left|f(z)\right|$

$$\left|\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty} 2k\,\partial(n,k)\, a_{2k} z^{2k}}{\sum_{k=2}^{\infty} (2k\xi+2k-1)\,\partial(n,k)\, a_{2k} z^{2k} + (\alpha-\beta)}\right\}\right| \le \gamma$$

Choosing z on real axis and allowing $z \to 1-$

$$\frac{\sum_{k=2}^{\infty} 2k\,\partial(n,k)\, a_{2k}}{\sum_{k=2}^{\infty} (2k\xi+2k-1)\,\partial(n,k)\, a_{2k} + (\alpha-\beta)} < \gamma$$

$$\sum_{k=2}^{\infty} [2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k) a_{2k} - \gamma(\alpha-\beta) \le 0$$

Thus the proof is complete.

Corollary:

If $f \in O_n(\alpha,\beta,\gamma,\xi)$ then

$$a_{2k} \le \frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}, \qquad k = 2,3,\ldots$$

with equality for

$$f(z) = z + \frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)} z^{2k}, \ k = 2,3,\ldots$$

3. GROWTH AND DISTORTION THEOREM:

Theorem 2:

If the function $f(z) \in O_n(\alpha,\beta,\gamma,\xi)$ then

$$|z| - \frac{\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)}|z|^4 \le |f(z)| \le |z| + \frac{\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)}|z|^4$$

The result is sharp for

$$f(z) = z + \frac{\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)} z^4$$

Proof: We have

$$f(z) = z + \sum_{k=2}^{\infty} a_{2k} z^{2k}$$

$$|f(z)| \le |z| + \sum_{k=2}^{\infty} a_{2k} |z|^{2k}$$

(2.1)

$$|f(z)| \le |z| + \frac{\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)}|z|^4$$

Similarly

$$|f(z)| \ge |z| - \sum_{k=2}^{\infty} a_{2k} |z|^{2k}$$

$$\ge |z| - \frac{\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)}|z|^4$$

Combining (2.1) and (2.2) we get the result.

Theorem 3:

If the function $f(z) \in O_n(\alpha,\beta,\gamma,\xi)$ then

$$1 - \frac{4\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)}|z|^3 \le |f'(z)| \le 1 + \frac{4\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)}|z|^3$$

The result is sharp for

$$f(z) = z + \frac{\gamma(\alpha-\beta)}{[4(1-\gamma\xi-\gamma)+\gamma]\partial(n,2)} z^4$$

4. RADIUS OF STARLIKENESS AND CONVEXITY:

Theorem 4: Let $f(z) \in O_n(\alpha,\beta,\gamma,\xi)$,then $f(z)$ is starlike in $|z| < R$ where

$$R = \inf_k \left\{ \frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{(2k+s-2)\gamma(\alpha-\beta)} \right\}^{\frac{1}{2k-1}}$$

, $k = 2,3,\ldots$

The estimate is sharp for the function

$$f(z) = z + \frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)} z^{2k}$$

for some k.

Proof: f is starlike of order $s, 0 \le s \le 1$ if

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)} \right) > s \qquad (3.1)$$

that is if

$$\left| z\frac{f'(z)}{f(z)} - 1 \right| \le 1-s \qquad (3.2)$$

This simplifies to

$$\sum_{k=2}^{\infty} \left| \frac{(2k+s-2)\, a_{2k} |z|^{2k-1}}{1-s} \right| \le 1$$

by (1.1) we have

$$a_{2k} \le \frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)} \ ,k \ge 2 \qquad (3.4)$$

Using (3.3) and (3.4) we get

$$|z|^{2k-1} \le \frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{(2k+s-2)\gamma(\alpha-\beta)}$$

thus,

$$|z| \le R = \inf_k \left\{ \begin{array}{l} \frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{(2k+s-2)\gamma(\alpha-\beta)} a_{2k}, \\ k = 2,3,\ldots \end{array} \right\}^{\frac{1}{2k-1}}$$

Theorem 5:

If $O_n(\alpha,\beta,\gamma,\xi)$ then $f(z)$ is convex of order c, $0 \le c < 1$ in $|z| < R$ where

$$R = \inf_k \left\{ \frac{(1-c)[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{2k(2k-2+c)\gamma(\alpha-\beta)} \right\}^{\frac{1}{2k-1}},$$

$k = 2,3\ldots$

estimate is sharp for

$$f(z) = z + \frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)} z^{2k} \text{ , for some } k.$$

Proof:

$f \in O_n(\alpha,\beta,\gamma,\xi)$ is convex of order c, $0 \le c \le 1$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \mathrm{c}$$

which is equivalent to

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1-c$$

That is

$$\left| \frac{\sum_{k=2}^{\infty} 2k(2k-1)\, a_{2k} z^{2k-1}}{1 + \sum_{k=2}^{\infty} 2k\ a_{2k} z^{2k-1}} \right| < 1-c$$

Using the arguments similar to theorem 4,We get

$$|z| < R = \inf_k \left\{ \frac{(1-c)[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{2k(2k-2+c)\gamma(\alpha-\beta)} \right\}^{\frac{1}{2k-1}}$$

5. EXTREME POINTS:

Theorem 6: Let $f_1(z) = z$,

$$f_k = z + \frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)} z^{2k} \ , \ k = 2,3\ldots$$

Then $f \in O_n(\alpha,\beta,\gamma,\xi)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \text{, where } \lambda_k \ge 0, \text{ and } \sum_{k=1}^{\infty} \lambda_k = 1.$$

Proof: Suppose

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

$$f(z)=\sum_{k=1}^{\infty}\lambda_k\left(z+\frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}z^{2k}\right)$$

$$f(z)=z+\sum_{k=1}^{\infty}\lambda_k\frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}z^{2k}$$

Now, $f(z)\in O_n(\alpha,\beta,\gamma,\xi)$ since

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}.$$

$$\frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}\lambda_k$$

$$\sum_{k=2}^{\infty}\lambda_k=1-\lambda_1\leq 1.$$

Conversely, suppose that $f\in O_n(\alpha,\beta,\gamma,\xi)$ then by (1.1)

$$a_{2k}\leq\frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)},\ k=2,3,\ldots$$

Setting

$$\lambda_k=\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}a_{2k},k=2,3,\ldots \quad (5.2)$$

And $\lambda_1=1-\sum_{k=2}^{\infty}\lambda_k$

We notice that, $f(z)=\sum_{k=2}^{\infty}\lambda_k f_k(z).\backslash$

Hence the result.

6. Hadamard Product

Theorem 7: Let $f,g\in O_n(\alpha,\beta,\gamma,\xi)$ then

$$f*g=z+\sum_{k=2}^{\infty}a_{2k}b_{2k}z^{2k}\in O_n(\alpha,\beta,\gamma,\xi)$$

for $f(z)=z+\sum_{k=2}^{\infty}a_{2k}z^{2k},\quad g(z)=z+\sum_{k=2}^{\infty}b_{2k}z^{2k}$

Where

$$\phi\geq\frac{2k\gamma^2(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]^2\partial(n,k)+\gamma^2(\alpha-\beta)(2k\xi+2k-1)}$$

Proof: $f,g\in O_n(\alpha,\beta,\gamma,\xi)$ and so

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}a_{2k}\leq 1 \quad (6.1)$$

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}b_{2k}\leq 1 \quad (6.2)$$

We need to find a smallest number ϕ such that

$$\sum_{k=2}^{\infty}\frac{[2k(1-\phi\xi-\phi)+\phi]\partial(n,k)}{\phi(\alpha-\beta)}a_{2k}b_{2k}\leq 1 \quad (6.3)$$

By Cauchy Schwarz inequality, we have

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}\sqrt{a_{2k}b_{2k}}\leq 1 \quad (6.4)$$

Thus, it is enough to show that

$$\frac{[2k(1-\phi\xi-\phi)+\phi]\partial(n,k)}{\phi(\alpha-\beta)}a_{2k}b_{2k}\leq \frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}\sqrt{a_{2k}b_{2k}}$$

That is

$$\sqrt{a_{2k}b_{2k}}\leq\frac{\phi[2k(1-\gamma\xi-\gamma)+\gamma]}{\gamma[2k(1-\phi\xi-\phi)+\phi]} \quad (6.5)$$

From (6.4)

$$\sqrt{a_{2k}b_{2k}}\leq\frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)} \quad (6.6)$$

Therefore in view of (6.5) and (6.6) it is enough to show that

$$\frac{\gamma(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}\leq\frac{\phi[2k(1-\gamma\xi-\gamma)+\gamma]}{\gamma[2k(1-\phi\xi-\phi)+\phi]}$$

This simplifies to

$$\phi\geq\frac{2k\gamma^2(\alpha-\beta)}{[2k(1-\gamma\xi-\gamma)+\gamma]^2\partial(n,k)+\gamma^2(\alpha-\beta)(2k\xi+2k-1)}$$

7. Closure Theorem

Theorem 8: Let $f_j \in O_n(\alpha,\beta,\gamma,\xi),\ j=1,2,\ldots$ then $g(z)=\sum_{j=1}^{l} c_j\left(z+\sum_{k=2}^{\infty} a_{2k,j}z^{2k}\right)$ where $\sum_{j=1}^{l} c_j = 1$ and $f_j(z)=z+\sum_{k=2}^{\infty} a_{2k,j}z^{2k}$.

Proof: We have

$$g(z)=\sum_{j=1}^{l} c_j\left(z+\sum_{k=2}^{\infty} a_{2k,j}z^{2k}\right)$$

$$g(z)=z\sum_{j=1}^{l} c_j+\sum_{j=1}^{l}\sum_{k=2}^{\infty} c_j a_{2k,j}z^{2k}$$

$$=z+\sum_{k=2}^{\infty}\sum_{j=1}^{l}\left(a_{2k,j}c_j\right)z^{2k} \quad (7.1)$$

$$=z+\sum_{k=2}^{\infty} e_k z^{2k} \quad (7.2)$$

Where, $e_k=\sum_{j=1}^{l} a_{2k,j}c_j$

Since $f_j \in O_n(\alpha,\beta,\gamma,\xi)$ by (1.1)

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}a_{2k,j}\le 1 \quad (7.3)$$

In view of (7.2), $g(z)\in O_n(\alpha,\beta,\gamma,\xi)$ if

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}e_k\le 1$$

Now,

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}e_k$$

$$=\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}\sum_{j=1}^{l}a_{2k,j}c_j$$

$$=\sum_{j=1}^{l}c_j\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}a_{2k,j}$$

$$\le\sum_{j=1}^{l}c_j \text{, using (7.3)}$$

$$=1$$

Thus, $g(z)\in O_n(\alpha,\beta,\gamma,\xi)$.

Theorem 9: Let $f,g\in O_n(\alpha,\beta,\gamma,\xi)$ then $h(z)=z+\sum_{k=2}^{\infty}\left(a_{2k}^2+b_{2k}^2\right)z^{2k}$ is in $O_n(\alpha,\beta,\phi,\xi)$ Where $\phi\ge 2\gamma$

Proof: $f,g\in O_n(\alpha,\beta,\gamma,\xi)$ and hence

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}a_{2k}^2\le\left[\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}a_{2k}^2\right]^2\le 1 \quad (8.1)$$

$$\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}b_{2k}^2\le\left[\sum_{k=2}^{\infty}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}b_{2k}^2\right]^2\le 1 \quad (8.2)$$

Adding (8.1) and (8.2) we get

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}\right]^2(a_{2k}^2+b_{2k}^2)\le 1 \quad (8.3)$$

We must show that $h\in O_n(\alpha,\beta,\phi,\xi)$ that is

$$\sum_{k=2}^{\infty}\left[\frac{[2k(1-\phi\xi-\phi)+\phi]\partial(n,k)}{\phi(\alpha-\beta)}\right]^2\left(a_{2k}^2+b_{2k}^2\right)\le 1 \quad (8.4)$$

In view of (8.3) and (8.4) it is enough to show that

$$\frac{[2k(1-\phi\xi-\phi)+\phi]\partial(n,k)}{\phi(\alpha-\beta)}\le\frac{1}{2}\frac{[2k(1-\gamma\xi-\gamma)+\gamma]\partial(n,k)}{\gamma(\alpha-\beta)}$$

This simplifies to

$$\phi\ge 2\gamma$$

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