GENERAL SOLUTION TO VAIDYA-TIKEKAR METRIC WITH CHARGED DISTRIBUTIONS ON SPHEROIDAL SPACE TIME

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ABSTRACT:

We are looking at the Vaidya-Tikekar metric which represents a three-dimensional space with time being constant, having charged distribution in a spheroidal super dense star. We address a general solution to Maxwell- Einstein's field equations in terms of hyper-geometric series. These models permit huge densities, radii of the order of few kilometers and maximum mass up to four times solar mass.

Keywords: Space-time, Neutron star, Einstein's field equations, Hypergeometric series

INTRODUCTION:

Stellar objects of spherical shape are generally electrically neutral in equilibrium. Even though the electric force of attraction prevalent in these objects prevents the collapse of a symmetrically spherical distribution of matter to a point singularity. This gravitational force of attraction is then balanced by the electrostatic force of repulsion as well as the pressure gradient within this stellar matter. These factors provide sufficient motivation for finding the interior sources for the Reissner-Nordstrom metric, which generally describes the space time of a static spherically symmetrical charge distribution. Reissner (1916) and Nordstrom (1918) found out a straight forward generalization of Schwarzschild exterior metric and called it as Reissner-Nordstrom metric. Subsequently many exact solutions of the coupled Einstein-Maxwell equations corresponding to charged distribution of spherical objects are reported.

A vast assemblage of stellar models having charge can be found in literature. Rainich (1925) has done a systematic study of electromagnetic fields in the background of general relativity and Papapetrou (1947) evaluated the equilibrium of charged spheres in the context of general relativity. Studies have been done that a fluid sphere having uniform density is more stable when it is charged as per Stenner (1973) and a matter distribution in a spherical environment retains its equilibrium if it is accompanied by an electric charge by Bonner (1960, 1965). In the case of static charged fluid spheres, a singularity free solution was obtained by Krori and Barua (1975) and was analyzed by Juvenicus (1976). Sah and Pant (1979) obtained a similar solution to Tolman Solution VI for spherically symmetric static charged fluid sphere. Cooperstock (1978) has done studies in perfect charged fluids which are in equilibrium and derived solutions explicitly for Maxwell-Einstein equations. V.O. Thomas and D.M. Pandya have obtained various analytic solutions by solving coupled Maxwell-Einstein equations, to spherical static symmetric systems having charge and they also made a stability analysis. Chang (1983) obtained flat interior solutions for charged dust distributions as well as charged fluids. Joshi (1993) discussed about the linear equation of state of charged anisotropic matter and found that the solutions to Maxwell- Einstein system of equations are critical in defining the formation of singularities and the influence of charge in determining the maximum mass for stars.

In this research paper, we have reached a new class of solutions for charged fluid distribution using hyper geometric series using 3D- spheroidal space-time. A particular option for electric field intensity and radial pressure are chosen so as not to violate the physical requirements and regularity conditions. In Sect. 2, we have defined the matter distribution and the metric considered for study. In Sect. 3, we have solved the Einstein's field equations using hyper geometric series and thus derived a general solution to Einstein's field equations and in Sect. 4 the physical plausibility and boundary conditions of this system was studied. In Sect. 5 particular solution for a specific value of K and its physical plausibility was studied in detail and, thereby progressing towards conclusion and discussion.

MATTER DISTRIBUTION ON SPHEROIDAL SPACE TIME:

Consider Tikekar and Vaidya's approach (1982) in representing the anisotropic charged matter distribution using the spheroidal spacetime metric as

$$ds^2 = -\frac{1-\frac{Kr^2}{R^2}}{1-\frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + sin\theta^2 d\varphi^2) + e^{\nu(r)}\,dt^2 \quad (1)$$

where R and K<1 are geometric parameters and $K = 1 - \frac{b^2}{R^2}$. *The metric variables of* $\frac{1-\frac{Kr^2}{R^2}}{1-\frac{r^2}{R^2}}$ *is related to the physical variables through Einstein's field equations which is given by* $R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^2}T_{ij}$, *where* R_{ij} *is the Ricci curvature tensor,* R *denotes scalar curvature,* g_{ij} *is the metric tensor and* T_{ij} *denotes the energy- momentum tensor.*

Considering the physical content of the space time to a charged fluid having associated energy momentum tensor as,

$$T_{ij} = \left(\rho + \frac{p}{c^2}\right)u_i u_j - \left(\frac{p}{c^2}\right)g_{ij} + \frac{1}{4\pi}\left[-F_{i\alpha}F^{i\alpha} + \frac{1}{4}g_{ij}F_{\alpha\beta}F^{\alpha\beta}\right] \quad (2)$$

where ρ represents matter density and p the fluid pressure.

F_{ij}, the anti-symmetric electromagnetic field tensor defined by

$$F_{ij} = \frac{(\partial A_j)}{\partial x_i} - \frac{(\partial A_i)}{\partial x_j} \quad (3)$$

and it satisfies the Maxwell's equations,

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \quad (4)$$

and

$$\frac{\partial}{x^{\alpha}}\left(F^{i\alpha}\sqrt{-g}\right) = 4\pi\sqrt{-g}\,J^i \quad (5)$$

where g denotes the determinant of g_{ij} and $J^i = \sigma u^i$ denotes the four-current vector, σ denotes the charge density and the unit four velocity field of matter is mentioned as $u^i = \left(0,0,0,e^{-\frac{\nu}{2}}\right)$.

Assuming the space time metric to be symmetric, it is evident that the only surviving term component of the electromagnetic field tensor is $F_{14} = F_{41}$.

Using this, the Maxwell's equation (5) for the spacetime metric (1) determines

$$F_{14} = e^{\frac{\nu+\lambda}{2}}\int_0^r 4\pi\sigma r^2 e^{\frac{1}{2}}\,dr \quad (6)$$

Electric field intensity E is defined as

$$E^2(r) = -F_{41}\,F^{41} \quad (7)$$

Hence from (5) and (6) it is clear that

$$4\pi\sigma = \frac{1}{r^2}\left[\frac{d}{dr}(r^2E)\right]e^{-\frac{\lambda}{2}} \quad (8)$$

The total charge contained in the sphere having radius r is given by

$$q(r) = 4\pi\int_0^r e^{\frac{\lambda}{2}}\,\sigma r^2 dr \quad (9)$$

And hence the electric intensity will be

$$E(r) = \frac{q(r)}{r^2} \quad (10)$$

From the energy momentum tensor ((2) defined, the Einstein's field equations reduce to the system of three equations given by

$$8\pi\rho + E^2 = -e^{-\lambda}\left[\frac{1}{r^2} - \frac{\lambda}{r}\right] + \frac{1}{r^2}$$
$$-8\pi p + E^2 = -e^{-\lambda}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] + \frac{1}{r^2}$$
$$-8\pi p + E^2 = -e^{-\lambda}\left[\nu' + \frac{\nu^2}{2} + \frac{\nu' - \lambda'}{2} - \frac{\nu'\lambda'}{2}\right] \quad (11)$$

Further substitution determining p, ρ and E^2 makes these three equations to

$$8\pi\rho = \frac{e^{-\lambda}}{2}\left[\frac{\nu'}{2} + \frac{\nu^2}{4} - \frac{\nu'\lambda}{4} - \frac{\nu' + 5\lambda}{4}\right]$$
$$8\pi p = \frac{e^{-\lambda}}{2}\left[\frac{\nu'}{2} + \frac{\nu^2}{4} + \frac{3\nu'\lambda}{2r} - \frac{\nu'\lambda}{4}\right] + \frac{e^{-\lambda} - 1}{r^2}$$
$$E^2 = \frac{e^{-\lambda}}{2}\left[\frac{\nu'}{2} + \frac{\nu^2}{4} - \frac{\nu' + \lambda}{2r} - \frac{\nu'\lambda'}{4}\right] + \frac{1 - e^{-\lambda}}{2r^2} \quad (12)$$

These three equations relate the four variables ρ, p, ν and E^2 since the assumption of spheroidal geometry for the space time fixes up e^{λ} as stated in (1). Specific system of this system of equations can only be obtained when one or more relation between these variables is available. Usually this relation is provided by the equation of state for the charged fluid. However, it is also possible to obtain specific solution by prescribing an adhoc relation, relating these variables. This is the approach which has been followed in a number of works mentioned earlier. The adhoc relation may be in the form of geometrical constraints or specific forms governing variations of either p, ρ or E^2 individually or their combinations. Hence it is necessary to examine the physical viability of the solution obtained. We have obtained two types of solution of Maxwell- Einstein equations.

Type I solution discussed follows a suitable path for the form of E^2 is given. A detailed explanation on physical viability of the solution for particular value of K is mentioned. Type II solution follows ON demanding the geometrical requirement that the spheroidal space-time be embedded in 5-dimensional flat space time.

GENERAL SOLUTION TO MAXWELL-EINSTEIN'S EQUATIONS:

A solution to Maxwell- Einstein's equations follows on prescribing

$$E^2 = \frac{\beta^2 r^2 e^{-\frac{\nu}{2}}}{R^4\left(1 - \frac{Kr^2}{R^2}\right)^2} \quad (13)$$

as a relation maintaining the variation of electrical field intensity responsible for the maintenance of equilibrium. Here β is a constant directly related to charge. From (13), it is very clear and evident that $E > 0$. Substituting (13) in (12), will help one to determine ν.

$$\frac{\beta^2 r^2 e^{-\frac{\nu}{2}}}{R^4\left(1 - \frac{Kr^2}{R^2}\right)^2} = \left[\frac{\nu}{2} + \frac{\nu^2}{4} - \frac{\nu'}{2r}\right]\left(1 - \frac{r^2}{R^2}\right)\left(1 - \frac{Kr^2}{R^2}\right)^{-1}$$
$$+\frac{1-K}{R^2}\left(1 - \frac{Kr^2}{R^2}\right)^{-1} - \frac{(1-K)r}{R^2}\left[\frac{\nu}{2} + \frac{1}{r}\right]\left(1 - \frac{Kr^2}{R^2}\right)^{-2} \quad (14)$$

By introducing z, ψ and defining these variables by the relation,

$z^2 = 1 - \frac{r^2}{R^2}$ and

$\psi = e^{\frac{\nu}{2}} = \frac{14\beta^2}{K(K-1)}$

the equation assumes the form of a linear second order differential equation given by

$$\left(1 - K + Kz^2\right)\frac{d^2\psi}{dz^2} - Kz\frac{d\psi}{dz} - +K(K-1)\psi = 0 \quad (15)$$

Defining an independent variable $, u^2. = \frac{K}{K-1}, z^2 \quad K < 0$ changes the differential equation (15) to the form

$$\left(1 - u^2\right)\frac{d^2\psi}{du^2} + u\frac{d\psi}{du} + (1-K)\psi = 0 \quad (16)$$

Further considering the new independent variable $x = u^2$ the differential equation (16) can be written in the form of a hyper-geometric equation as

$$x(1-x)\frac{d^2\psi}{dx^2} + \frac{1}{2}\frac{d\psi}{dx} + \frac{1-K}{4}\psi = 0 \quad (17)$$

The function ψ which satisfies the above equation can be equated to

$$\psi = e^{\frac{\nu}{2}} = AF\left[\frac{-1+\sqrt{2+K}}{2}, \frac{-1-\sqrt{2+K}}{2}, \frac{1}{2}, x\right]$$
$$+Bx^{\frac{1}{2}}F\left[\frac{\sqrt{2-K}}{2}, \frac{-\sqrt{2-K}}{2}, \frac{3}{2}, x\right] \quad (18)$$

where $F[a,b,c,x]$ is the hyper-geometric function with its arguments A and B are arbitrary constants. This (18) admits to be the general solution of the above differential equation (17).

EXACT SOLUTION FOR $K = -14$ AND ITS PHYSCAL PLAUSIBILITY:

As a special case to strengthen the above solution obtained, we are considering a particular value of K to be K=-14.

For K=-14, the closed form solution becomes,

$$e^{\frac{\nu}{2}} = \frac{\beta^2}{15} + A\left(1 - \frac{14}{15}z^2\right)^{\frac{3}{2}}\left(1 - 6\frac{14}{15}z^2\right) + Bz\left(1 - \frac{8}{3}\frac{14}{15}z^2 + \frac{8}{5}\left(\frac{14}{15}z^2\right)^2\right) \tag{19}$$

Explicit expressions for matter density and fluid pressure were formulated. The solution will describe a space time of a physically viable distribution of charged fluid if it complies with the requirements such as $\rho > 0, p > 0$ and $\rho - 3p > 0$.

The implication of these conditions at the center was examined and the density at the center came out to be, $8\pi\rho(0) = \frac{45}{R^2}$ and the positivity of density at the center is evident from the expression.

The condition p(0)>0 will be satisfied when any of the following condition hold true.

$-5.08A + 0.42B < \beta^2 < 1.19A + 1.43B$ or

$1.19A + 0.42B < \beta^2 < -0.58A + 0.42B$

The condition $\rho - 3p > 0$ implies that $\frac{15}{R^2}\left[\frac{-875.31A+416B-450\beta^2}{89.08A+107B-75\beta^2}\right] \geq 0$ which gives the relation

$-1.95A + 0.92B \geq \beta^2$ or $-1.95A + 0.92B \leq \beta^2$

If the distribution extends up to a finite radius $a < R$, the interior metric should continuously match with the exterior metric given by Reissner-Nordstorm metric given by,

$$ds^2 = -\left[1 - \frac{2m}{a} + \frac{q^2}{a^2}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 d\varphi^2 + \left[1 - \frac{2m}{a} + \frac{q^2}{a^2}\right] dt^2 \tag{20}$$

At the boundary r=a, the fluid pressure vanishes and hence

$e^{\nu(a)} = e^{-\lambda(a)} = 1 - \frac{2m}{a} + \frac{q^2}{a^2}$

and

$$\frac{\beta^2\left(1 + 13\frac{a^2}{R^2}\right)}{1 + 14\frac{a^2}{R^2}} + \frac{A\left(1 + 14\frac{a^2}{R^2}\right)^{\frac{1}{2}}}{3\sqrt{15}}\left[59 - 382\frac{a^2}{R^2} + 392\frac{a^4}{R^4}\right] + B\frac{17}{765}\left(1 - \frac{a^2}{R^2}\right)^{\frac{1}{2}}\left[-19 - 784\frac{a^2}{R^2} + 1568\frac{a^4}{R^4}\right] = 0$$

From the above equations, A and B can be determined in terms of β^2 and $\frac{a^2}{R^2}$.

The total charge of the sphere will be

$$q^2 = \alpha^2\beta^2[R^4\left(1 + 14\frac{a^2}{R^2}\right)^2 \begin{bmatrix} \beta\frac{2}{15} + A\left(1 - \frac{14}{15}z_a^2\right)^{\frac{3}{2}}\left(1 - 6\frac{14}{15}z_a^2\right) \\ +Bz_a\left(1 - \frac{8}{3}\frac{14}{15}z_a^2 + \frac{8}{5}\left(\frac{14}{15}z_a^2\right)^2\right] \end{bmatrix} \tag{21}$$

It is evident that the power switches off when β becomes 0 and hence the solution will degenerate to that of an uncharged fluid sphere. And the mass of the fluid sphere can be determined from the boundary condition as

$$\frac{2m}{a} = \frac{15\frac{a^2}{R^2}}{1 + 14\frac{a^2}{R^2}} + \frac{q^2}{a^2} \tag{22}$$

All the variables in determining the mass of the fluid sphere are familiar to us from different equations generated. We have also studied the variation of ρ, p and $\rho - 3p$ using numerical procedure and the condition on $\frac{dp}{d\rho}$ as these are evident from the tabular form hence generated.

SCHEME FOR COMPUTATION OF MASS AND SIZE OF THE FLUID SPHERE:

The scheme for the computation of mass and size of the charged fluid sphere can be evaluated by defining a new parameter, μ as the ratio of density at the boundary to that at the center.

$$\mu = \frac{\rho(a)}{\rho(0)} \qquad (22)$$

TABLE 1: MASS FOR $\beta^2 = 2.0$

$\frac{a}{R}$	A	B	R	a	$\frac{m}{M_\theta}$	Q
0.14	-1.2993	-5.9919	89.69	12.68	1.01	0.300
0.20	-1.3902	-4.3352	76.61	15.32	2.01	0.627
0.24	-1.4123	-3.0813	67.43	16.52	2.76	0.901
0.28	-1.3949	-2.0615	60.60	17.14	3.31	1.127
0.31	-1.3534	-1.2254	55.30	17.49	3.74	1.317
0.34	-1.2965	-0.5294	51.07	17.69	4.07	1.480
0.37	-1.2297	0.0559	47.59	17.81	4.33	1.622
0.40	-1.1564	0.5513	44.68	17.87	4.55	1.749
0.42	-1.0790	0.9719.	42.20	17.90	4.72	1.864
0.44	-0.9990	1.3292	40.06	17.92	4.87	1.969
0.46	-0.9177	1.6321	38.19	17.91	4.99	2.066
0.48	-0.8359	1.8875	36.54	17.90	5.10	2.157

[a] The mass m for the class of charged fluid spheres having $\beta^2 = 2.0$ for different values of $\frac{a}{R}$, A and B and the same is tabulated.

TABLE 2: NUMERICAL PROCEDURE.

$\frac{a}{R}$	$p(0)$	$\rho(0)$	$\rho(0) - 3p(0)$
0.1414	0.00031	0.00527	0.00463
0.2000	0.00083	0.00683	0.00516
0.2449	0.00158	0.00831	0.00515
0.2828	0.00260	0.00964	0.00443
0.3162	0.00398	0.01072	0.00274
0.3464	0.00583	0.01141	-0.00026
0.3741	0.00836	0.0115	-0.00521
0.4000	0.01192	0.01061	-0.01322
0.4242	0.01728	0.00797	-0.02659
0.4472	0.02632	0.00171	-0.05093
0.4690	0.04522	-0.01437	-0.10482
0.4898	0.11389	-0.0802	-0.30798

[b] The values of $\rho, p, \rho - p$ and $\rho - 3p$ evaluated at the center using numerical procedure.

RESULTS AND DISCUSSION:

Tikekar and Vaidya (1982) model of super dense fluid spheres with densities of fluid matter content of $10^{14} - 10^{16}\mathrm{gmcm}^{-3}$ in range is usually developed during the last stages of stellar development. Here we assumed that at the boundary (when r becomes a), the density of the star becomes $2 \times 10^{14}\,\mathrm{gmcm}^{-3}$ and this particular value corresponds to that of neutron star.

In this criteria defined, $z_a^2 = 1 - \frac{a^2}{R^2}$, from the given expression is clear that μ can be defined for all values of $\frac{a}{R}$ in terms of β. Only those values will be physically permissible for which $0 < \mu < 1$. Subsequently if the surface density ρ(a) and μ are specified, R can be determined from the equation, $\rho(0) = \frac{\rho(a)}{\mu} = \frac{45c^2}{8\pi G R^2}$ in terms of μ, β and $\rho(a)$. From these a can be found and (21) helps us to decide q in terms of β and $\frac{a}{R}$ and the total mass can be determined from (22).

For estimation we have assumed the value of ρ(a) as $2 \times 10^{14}\ \mathrm{gm\,cm}^{-3}$ for the surface density of matter. Hence we have arrived at the values of A and B, the curvature parameter R, the boundary radius a, the total charge q and the mass m for the class of charged fluid spheres having $\beta^2 = 0$ for different values of $\frac{a}{R}$ and the same is tabulated in table 1. Strong and weak conditions including pressure and density for different values of $\frac{a}{R}$ is tabulated in the next table. From the second table it is clear that, all models with $\frac{a}{R} \leq 0.31$ comply with the requirement $\rho(0) \geq 3p(0)$ in addition to $\rho(0) > 0$ and $p(0) > 0$. However the models with $\frac{a}{R} > 0.46$ are not physically viable as the requirement of weak condition $\rho(0) - p(0) > 0$ is violated. Hence from table 1 it is noted that a charged spherical fluid could hold a maximum mass of 3.74 which corresponds to $\frac{a}{R} = 0.31$.

In conclusion the static spheroidal space-time can be expected to describe the interior of superdense fluid sphere in equilibrium accompanied by presence of charge.

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