STATIC GENERAL SOLUTION TO EINSTEIN'S FIELD EQUATIONS

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ABSTRACT:

We consider the Vaidya Tikekar metric; 3- dimensional space with time (t) being constant, in a super dense star which is spheroidal. We report a static general solution (in terms of hyper-geometric series) to Einstein's field equations using Vaidya-Tikekar metric.

Keywords: Einstein's field equations, Space-time, Hypergeometric series

INTRODUCTION:

General relativity theory incorporates gravity as a phenomenon intrinsically related with the geometry of space-time and succeeds in arriving at a deeper understanding of the space-time associated with distribution of physical fields of cosmological and astrophysical significance. Especially after the discovery of pulsars which are believed to be rotating neutron stars, there is lot of interest in using general relativity theory to study the interior structure of the space-times of superdense configuration of matter with matter densities exceeding the density of nuclear matter. Use of general relativity theory in understanding the final stages of evolution of a star such as neutron star and black hole is considered to be unavoidable. The effective mass of gravitational field as obvious from the Einstein's field equations of general relativity enforces constraints in obtaining simple exact solutions which may function as models of relativistic stars. Lack of dependable knowledge about the properties of central core area of comparatively compact stars is another hindrance which warrants of a general nature. Einstein's field equations in the theory of general relativity represents a collection of partial differential nonlinear equations in four separate variables. This arduous system of nonlinear partial differential equations is hard to integrate, despite of it being considered as a self-coupled integral equation. Accordingly, it is desired to have some analytic solutions at hand, which may serve as easily surveyable models for these stars [Tolman (1939)]. Even then, as one cannot have believable knowledge about the behaviour of matter at the core region of neutron star under extreme conditions, we can warranty only the most general assumptions. If similar closed form solutions [Tolman (1939); Adler (1974); Leibovitz (1969); Buchdahl (1959)] adhere to certain general fundamental properties expected from fluids at ultra-high masses and pressure; then it will be of astrophysical significance.

Tikekar and Vaidya (1982) have established that the space-times with t being constant having the geometry of a 3D-spheroid characterized by two parameters K measuring the oblateness and R showing the spherical nature of the spheroid are useful in developing easily surveyable relativistic design for super dense stars such as neutron stars. It is demonstrated that these space-times can be utilized to establish static models characterizing the field of gravity in the interior of superdense condensation matter like neutron stars and white dwarfs. The physical soundness of the class of models by Tikekar and Vaidya was examined by Knutsen (1988) and concluded that these models are stable with respect to infinitesimal radial beats. Tikekar (1990) again reported another class of models with above geometry. References shows that only a limited number of analytic closed form solutions of Einstein's field equations for static spherical distributions of material can be useful as easily significant modes for superdense stars, it is necessary to investigate the suitability of other particular classes of models in this set up.

This paper deals with the study of spheroidal space time and its suitability to represent the interior of compact fluid spheres in equilibrium. The space times are characterized by two curvature parameters R and K. The requirement that the space time of a matter distribution in equilibrium be spheroidal determines the law of variation of density of matter in the configuration and the problem of solving a second order linear differential equation. Maharaj and Leach (1996) and Mukherjee et al. (1997) have discussed methods for solving this differential equation. We have discussed two methods for obtaining general solution to Einstein's field equations one of which is similar to the one given by Maharaj and Leach (1996). Our other method consists of converting this differential equation to hyper geometric function and hence reaching out with an exact general solution to Einstein's field equation.

SPHEROIDAL SPACE TIME:

We consider a four-dimensional Euclidean space with metric

$$\mathbf{d\sigma^2 = dx^2 + dy^2 + dz^2 + dt^2} \qquad (1)$$

A three dimensional immersed in the four-dimensional Euclidean space will have the Cartesian equation:

$$\frac{\mathbf{w^2}}{\mathbf{b^2}} + \frac{\mathbf{x^2 + y^2 + z^2}}{\mathbf{R^2}} = \mathbf{1} \qquad (2)$$

The parameterization,

$x = R\sin\alpha\cos\theta\cos\varphi$

$y = R\sin\alpha\sin\theta\sin\varphi$

$z = R\sin\alpha\cos\theta$

$w = b\cos\theta$

of the three spheroid leads to

$$\mathbf{d\sigma^2 = (R^2\cos^2\alpha + b^2\sin^2\alpha)d\alpha^2 + R^2\sin^2\alpha(d\theta^2 + \sin^2\theta\, d\varphi^2)} \qquad (3)$$

as the metric on the three-dimensional spheroid.

Introducing a new variable $r = R\sin\theta$ the metric assumes the form,

$$\boldsymbol{d\sigma^2 = -\frac{1-\frac{Kr^2}{R^2}}{1-\frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)} \qquad (4)$$

where $K = 1 - \frac{b^2}{R^2}$ represents the space-time in the interior of a spherical distribution of matter at rest.

The form of this metric indicates that

1. The spheroidal 3-space is spherically symmetrical.
2. The metric is regular and positive definite at all points $r < R$.
3. For $K = 1$, the spheroidal 3-space degenerates into a flat space.
4. For $K = 0$, the spheroidal 3-space degenerates into a sphere.

DISTRIBUTION OF MATTER IN SPHEROIDAL SPACE TIME:

Tikekar and Vaidya have shown that the solution of Einstein's filed equations describing matter distribution on static spheroidal space time are highly relevant in describing the interior pace time of superdense stars. A relativistic model of a superdense star has been reported by Tikekar (1990) and certain general aspects of such models have been investigated by Knutsen (1988). Maharaj and Leach (1996). We shall discuss here certain general aspects of geometrical and physical significance of matter distribution on spheroidal space-time and general methods for obtaining models of superdense spherical configurations of matter in equilibrium on the background of spheroidal space-times. A 'C' program useful for obtaining information about various parameters of physical relevance associated with models corresponding to specific choices of certain parameters of geometrical and physical relevance will also be given. Looking at Tikekar and Vaidya's (1982) approach, we consider the static spherically symmetric space-time with the metric,

$$d\sigma^2 = -\frac{1-\frac{Kr^2}{R^2}}{1-\frac{r^2}{R^2}}dr^2 - r^2\left(d\theta^2 + sin^2\theta\, d\varphi^2\right) + e^{\nu(r)}dt^2 \quad (5)$$

Considering the physical content of the space time to an ideal fluid having associated energy momentum tensor as,

$$T_{ij} = \left(\rho + \frac{p}{c^2}\right)u_i u_j - \left(\frac{p}{c^2}\right)g_{ij} \quad (6)$$

where ρ represents matter density and p the fluid pressure. Representing the unit four velocity field of matter mentioned as $u_i = \left(0,0,0,,e^{-\frac{\nu}{2}}\right)$,

Einstein's field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^2}T_{ij} \quad (7)$$

reduces to the system of three equations given by

$$8\pi\rho = \frac{\frac{3(1-K)}{R}\left[1-\frac{\frac{K}{3}r^2}{R^2}\right]}{\left[1-\frac{Kr^2}{R^2}\right]^2} \quad (8)$$

$$8\pi p = \left[\frac{\nu'}{r} + \frac{1}{r^2}\right]\frac{\left[1-\frac{r^2}{R^2}\right]}{\left[1-\frac{Kr^2}{R^2}\right]} - \frac{1}{r^2} \quad (9)$$

and

$$c\left[1-\frac{Kr^2}{R^2}\right]\left[\nu'' + \frac{\nu'^2}{2} - \frac{\nu'}{r}\right] - \frac{1-K}{R^2}(r\nu' + 2) + \frac{2(1-K)}{R^2}\left[1-\frac{Kr^2}{R^2}\right] = 0 \quad (10)$$

Here and in what follows an overhead prime mentions differentiation with respect to radial variable r. In our proposal the prevailing choice of state of matter is replaced with the option of the spheroidal geometry which demonstrate the rate of change with respect to r. Equation (8) displays that the density of the fluid is figured out by the curvature of the physical 3-space. The field equation (9) presents the variation of pressure with r when ν is chosen to satisfy Equation (10). It is shown by Tikekar (1990), that the relativistic condition for hydro static equilibrium is:

$$\frac{1}{c^2}\left(\frac{dp}{dr}\right) = -\frac{\left(\rho + \frac{p}{c^2}\right)}{r^2}\left[\frac{m(r) + \frac{4\pi Gp}{c^4}r^3}{\left(1-\frac{2m}{r}\right)}\right] \quad (11)$$

This usually replaces the field equations (10) with the explicit form to,

$$\frac{1}{c^2}\left(\frac{dp}{dr}\right) = -\frac{\left[1-\frac{Kr^2}{R^2}\right]}{\left[1-\frac{r^2}{R^2}\right]}\left[\frac{4\pi Gpr}{c^4} + \frac{(1-K)r}{2R^2\left[1-\frac{Kr^2}{R^2}\right]}\right]\left(\rho + \frac{p}{c^2}\right) \quad (12)$$

This law points out that the pressure gradient coupled with the repulsive force make up for the gravitational force of attraction of matter and thus establishes equilibrium. We shall examine how the law of variation of the density given by Equation (8) facilitates us to assess the mass and the radius of the configuration

STATIC GENERAL SOLUTION OF EINSTEIN'S FIELD EQUATIONS:

Adopting new variable ψ and z^2 defined to be as,

$$\psi = e^{\frac{\nu}{2}}$$
$$z^2 = 1 - \frac{r^2}{R^2} \qquad (13)$$

and substituting into the second order, nonlinear ordinary differential equation (10), resulting in a second order linear differential equation of the form,

$$(1 - K + Kz^2)\frac{d^2\psi}{dz^2} - Kz\frac{d\psi}{dz} - +K(K-1).\psi = 0 \quad (14)$$

Defining an independent variable $u^2 = \frac{K}{K-1}z^2,\ K < 0$ changes the differential equation (14) to the form

$$(1 - u^2)\frac{d^2\psi}{du^2} + u\frac{d\psi}{du} + (1 - K)\psi = 0 \qquad (15)$$

used by Tikekar and Vaidya (1982). They had derived a series solution for this equation considering the form $\psi = \sum A_k u^k$ which leads to a recurrence relation for the coefficients A_k as

$$(n+1)(n+2)A_{n+2} = (n^2 - 2n + K - 1)A_n.$$

If the parameter K has value such that the equation, $n^2 - 2n + K - 1 = 0$ admits integral values of n as solutions, either of the two sets $(A_0, A_2, A_4, \dots)$ *or* $(A_1, A_3, A_5 \dots)$ contains finite number of elements and the corresponding terms in the solution series constitute a finite polynomial. It can be verified that, for it to be in the range $K < 1$, the simplest value of K *is* – 2 which corresponds to $n = 3$.

For $K = -2$, Tikekar and Vaidya had derived the following solution,

$$\psi = A_0\left(1 - \frac{3}{2}u^2 + \frac{3}{8}u^4 \dots\right) + A_1 u\left(1 - \frac{2}{3}u^2\right)$$

Observing the infinite series with A_0 as the coefficient, the closed form solution for $K = -2$ will be

$$exp\left(\frac{\nu}{2}\right) = \psi = Az\left(1 - \frac{4}{9}z^2\right) + B\left(1 - \frac{2}{3}z^2\right)^{\frac{3}{2}}$$

Closed-form solutions of Equation (11) have also been obtained for $K = -7, -14, -23 \dots$ But this method can give solution only to certain values of K which satisfies the recurrence relation. There arises the importance of an exact general solution applicable for all values of K.

Further considering the new independent variable $x = u^2$, the differential equation (15) can be written in the form of a hyper-geometric equation as,

$$x(1-x)\frac{d^2\psi}{dx^2} + \frac{1}{2}\frac{d\psi}{dx} + \frac{1-K}{4}\psi = 0 \qquad (16)$$

The function ψ which satisfies the above equation can be equated to,

$$\psi = e^{\frac{\nu}{2}} = AF\left[\frac{-1+\sqrt{2+K}}{2}, \frac{-1-\sqrt{2+K}}{2}, \frac{1}{2}, x\right] + Bx^{\frac{1}{2}}\,F\left[\frac{\sqrt{2-K}}{2}, \frac{-\sqrt{2-K}}{2}, \frac{3}{2}, x\right] \qquad (17)$$

where $F[a, b, c, x]$ is the hyper-geometric function with its arguments A and B which are arbitrary constants. The closed form solutions that can be obtained from (17) which can be divided into two classes based on the values of K. The solution ψ for the equation (17) for both the cases of K can be determined.

Case 1: $K = 2 - (2n^2 - 1),\ n = 2, 3, 4 \dots$

$$\psi = e^{\frac{\nu}{2}} = \mathrm{AF}\left[\frac{-1+\sqrt{2-K}}{2}, \frac{-1-\sqrt{2-K}}{2}, \frac{1}{2}, x\right] + Bx^{\frac{1}{2}}\,(1-x)^{\frac{3}{2}}\,F\left[\frac{3+\sqrt{2-K}}{2}, \frac{3-\sqrt{2-K}}{2}, \frac{3}{2}, x\right] \qquad (18)$$

Case 2: $K = 2 - 4n^2,\ n = 1, 2, 3, 4 \dots$

$$\psi = e^{\frac{\nu}{2}} = A(1-x)^{\frac{3}{2}}\,F\left[\frac{2+\sqrt{2-K}}{2}, \frac{2-\sqrt{2-K}}{2}, \frac{1}{2}, x\right] + Bx^{\frac{1}{2}}\,F\left[\frac{\sqrt{2-K}}{2}, \frac{-\sqrt{2-K}}{2}, \frac{3}{2}, x\right] \qquad (19)$$

The above solution can be used in general for all K < 0 as far as numerical calculations are concerned. As we have reached the solution for the Einstein's field equation without making any assumption on the equation of state for its matter content, it is required to examine the physical plausibility of the solution thus obtained.

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