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USING NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS AND LINEARIZED NAVIER-STOKES EQUATIONS VISCOUS NEWTONIAN FLUID MODEL FOR BLOOD VESSEL WALLS

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ABSTRACT:

This article describes a mathematical model of the circulatory system for the cardiovascular system and provides a basic framework for the mathematical representation of cumulative medical parameters such as total vascular area about numerical solution of the Navier-Stokes equations and linearized Navier-Stokes equations viscous Newtonian fluid model for blood vessel walls, blood volume, self-regulation, and effects on the upper and inner heart. In mathematical terms, linear dependencies, differential, integral and differential equations are used.

Keywords: linear dependence, integral-differential equations, logical-dynamic equations, general vascular zone, self-regulation, influence on the upper and working heart, medical parameters.

INTRODUCTION

The tone of the mathematical model of the degree of relation to the cardiovascular system and the circulatory system of three-dimensional model of the flow of an incompressible viscous Newtonian fluid, which is related to time, is considered in the form of several specific areas of the cardiovascular system. When the experimental results, which reflect the volume tone and fluid dependence (blood fluidity), pressure and heart rhythm are reflected in the myocardial characteristics (in chronotope relations, dependence on velocity), the relationship between phases is shortened and weakened. Other pathologies require surgical intervention, for example, aneurysms, malformations.

METHODS

Assume that the approximation to the solution of the system of equations (1.1) $u_k\, p_k$ at time $t_k = k\Delta t,\ k = 1, ..., n$ designed and required to find the unknown $u_{n+1},\ p_{n+1}$, with t_{n+1}. Approximating the time derivative with the second order accuracy in time t_{n+1}, we get the following scheme:

$$\begin{cases} \frac{1}{2\Delta t}(3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^{n} + \boldsymbol{u}^{n-1}) + w \cdot \nabla \boldsymbol{u}^{n+1} - \nu\nabla \boldsymbol{u}^{n+1} + \nabla \boldsymbol{p}^{n+1} = f^{n+1} \\ div\ \boldsymbol{u}^{n+1} = 0 \\ \boldsymbol{u}^{n+1}\left|\Gamma_{in} = \boldsymbol{u}_{in}^{n+1},\ \boldsymbol{u}^{n+1}\right|\Gamma_{in} = 0, \quad \left(-\nu\frac{\partial \boldsymbol{u}^{n+1}}{\partial \boldsymbol{n}} + \rho^{n+1}\boldsymbol{n}\right)\Big|\Gamma_{out} = \phi^{n+1} \end{cases} \quad (1.2)$$

The choice of the expression for w allows either to linearize the convective term of the Navier-Stokes equations, or to preserve its nonlinearity. In the first case, the value of **w** is extrapolated from the solutions from the two previous time steps with the second order of accuracy:

$$\mathbf{w} = (2\ \mathbf{u}^{n} - \mathbf{u}^{n-1}), \quad (1.3)$$

thus, a linear system of differential equations is obtained, known as the Ozein problem. In the second case, the value of **w** is equal to the value of the speed at the $(n + 1)$ the time step:

$$\mathbf{w} = \mathbf{u}^{n+1}, \quad (1.4)$$

The problem (1.1) turns out to be nonlinear and we use the Newton-Krylov method to solve it. The discretization of (1.1) under the condition (1.2) leads to the need to solve the Ozein problem at each time step. Consequently, the main part of computational resources will be spent on the calculation of this problem when implementing a two-scale model of fluid flow. Therefore, special attention will be paid to numerical methods for solving the linearized Navier-Stokes equations.

In general, the Ozein problem in the region Ω with the Dirichlet condition at the flow boundary Γ_{in}, the adhesion condition on the channel wall Γ_0 and the Neumann condition at the output boundary Γ_{out} has the following form:

$$\begin{cases} \alpha \boldsymbol{u} - \boldsymbol{u}\Delta u + (w \cdot \nabla)\Delta u + \nabla p = f \\ \qquad\qquad\qquad\qquad in\Omega \\ div\ \boldsymbol{u} = 0 \\ \boldsymbol{u}|\Gamma_{in} = g, \quad u|\Gamma_0 = 0, \\ \left(v\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - p\boldsymbol{n}\right)|\Gamma_{out} = \phi, \\ \boldsymbol{u_0} = h \end{cases} \quad (1.5)$$

The weak formulation of the equations (1.5) consists in finding a function $\mathbf{u} \in \mathbf{V}_g = \{u \in H^1(\Omega): u \mid \Gamma_{in} = g, u \mid \Gamma_0 = 0\}$ and $p \in Q$, satisfying the equations:

$$\begin{cases} \alpha(\boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla u, \nabla v) + ((w\cdot\nabla)u, v) - (p, \operatorname{div} v) = (f^*, v) - \int_{\Gamma_{out}} v\cdot\phi, \forall v \in V, \\ (\operatorname{div} u, q) = 0, \forall_q \in Q. \end{cases}$$

(1.6)

For the problem (1.1) - (1.2), we used the notation:

$$\alpha = 3(2\Delta t)^{-1}, f^* = f^{n+1} - (4u^n - u^{n-1})(2\Delta t)^{-1}, g = u_{in}^{n+1}, h = 2u_n - u_{n-1}.$$

The dissertation uses the finite element method to discretize the system of equations (1.5).

DISCRETIZATION OF THE OZEIN PROBLEM:

We introduce finite-dimensional spaces for velocity $V_h \subset V_g$ and for pressure $\mathbb{Q}_h \subset L^2(\Omega)$, approximating the spaces V_g and $L^2(\Omega)$, respectively. Let $\mathbb{V}_h^0$ put $(\psi, \varphi)_{\mathbb{V}} = (\nabla\psi, \nabla\varphi)$, and we believe that the conditions of elasticity, continuity and sustainability:

$$\alpha_1\|v_h\|_{\mathbb{V}}^2 \le \alpha_h(v_h, v_h), \quad a(v_h, u_h) \le \alpha_2\|v_h\|_{\mathbb{V}}\|u_h\|_{\mathbb{V}} \quad \forall v_h, u_h \in \mathbb{V}_h^0 \quad (2.1)$$

$$\Upsilon_1^2\|q_h\|^2 \le \sup_{v_h \in \mathbb{V}_h^0} \frac{(q_h, \operatorname{div} v_h)^2}{\|v_h\|_{\mathbb{V}}^2} \quad \forall\, q_h \in \mathbb{Q}_h \quad (2.2)$$

$$(q_h, \operatorname{div} v_h) \le \gamma_2\|q_h\|\|v_h\|_{\mathbb{V}} \quad \forall\, q_h, p \in \mathbb{Q}_h, \quad v_h \in \mathbb{V}_h^0 \quad (2.3)$$

with positive constants $\alpha 1$, $\alpha 2$, $\gamma 1$, $\gamma 2$, independent of the computational grid. The condition (2.2) is also known as the LBB-inequality or inf-sup inequality.

The finite element solution of problem (2.4) is the functions $u_h \in \mathbb{V}_h$ and $p_h \in \mathbb{Q}_h$, satisfying the equations:

$$a(u_h, v_h) - (p_h, \operatorname{div} v_h) + (q_h, \operatorname{div} u_h) = (f_h^*, v_h) - \int_{\Gamma_{out}} v_h \cdot \phi \quad \forall v_h \in \mathbb{V}_h^0, \forall v_h, \ q_h \in \mathbb{Q}_h$$

where

$$a(u_h, v_h) = \alpha(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + (w\cdot\nabla u_h, v_h) \quad (2.5)$$

Let is choose the bases $\{\phi i, i = 1, ..., N\mathbb{V}h\}$ and $\{\psi i, i = 1, ..., N\mathbb{Q}\}$ in the spaces 0 and $q\hbar$, where $\mathbb{V}_h^0$, $=\dim(\mathbb{V}_h^0,)$ and $N\mathbb{Q}_h$, $=\dim(\mathbb{Q}_h,)$. We will look for a solution in the form of a linear combination of basis vectors:

$$u_h(x) = u_h^g + \sum_{i=1}^{N\mathbb{V}_h} u_i \varphi_i(x), \quad p_h(x) = \sum_{i=1}^{N\mathbb{Q}_h} p_i \psi_i(x),$$

where $u_h^g \in \mathbb{V}_h$ is an arbitrary vector function. Then the problem (2.5) is equivalent to a system of linear equations with a Saddle point with respect to unknown vectors of coefficients:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} \quad (2.6)$$

where

$$(U)_i = u_i, \quad (P)_i = p_i,$$

$$(F)_i = (f_h^*, \varphi_i) - \int_{\Gamma_{out}} \varphi_i \cdot \phi - a(u_h^g, \varphi_i),$$

$$(G)_i = (\text{div } u_h^g, \psi_i)$$

$$A_{ij} = a(\varphi_j, \varphi_i), \quad B_{ij} = -(\text{div } \varphi_j, \psi_i),$$

In this paper, quadratic and linear basis functions are used to approximate the velocity and pressure, respectively (P2-P1 elements). The sampling is carried out using the ani3D software package [1].

NUMERICAL SOLUTION OF A SYSTEM OF LINEAR EQUATIONS WITH A SADDLE POINT:

The system of equations (2.5) in the dissertation is proposed to be solved by the method of bi-conjugate gradients [2] with a block triangular determinant [3,4]:

$$P = \begin{pmatrix} \hat{A} & B^T \\ O & -\hat{S} \end{pmatrix} \quad (2.6)$$

The block A ^ is the re-determinant of the matrix A. To construct it, you can use multigrid methods [13] or domain decomposition methods [1, 3, 4]. These algorithms are effective for a sufficiently large range of viscosity ν and scale well. In practice, when using the preconditioned (2.6), the inverse matrix A^-1 is required. It can be given implicitly by means of an inaccurate solution of systems of linear equations of the form$A$ A_y=x, where x is a certain vector. In fact, it is not necessary to know the components of the matrix A^-1 explicitly, it is enough to be able to calculate the result of multiplying the vector by it. For this purpose, in this paper we will use the V-cycle of the multigrid method. A description of how the loop of a multigrid method sets the determinant A^-1 can be found in section 2.5 of [1].

Where S = BA $^{-1}B^T$. The matrix S is not sparse. Moreover, its construction explicitly requires the inversion of the matrix A. Thus, the calculation of the over conditioner for S^ is not a standard task. In practice, only the inverse matrix is of interest, the following determinant is:

$$\hat{S}^{-1} \coloneqq \widehat{M}_p^{-1} A_p L_p^{-1} \quad (2.7)$$

$(\mathcal{M}_p)_{ij,} = (\psi_j, \psi_i)$ is the mass matrix for pressure (similarly, we will meet I $(\mathcal{M})_{,} = (\varphi, \varphi)$, - mass matrix for velocities);

M^ $_p$$^{-1}$ approximate solution of a system of equations with a mass matrixM_p. The matrices A_p and L_p approximate the convection-diffusion and Laplace operators in Q_h, respectively, and require taking into account the boundary conditions for pressure (explicitly or implicitly).

If Q_h approximates the pressure space and $Q_h \in H^1(\Omega)$, one can use a discretization of the Poisson problem for pressure with Neumann boundary conditions to determine L_p:

$$(L_p)_{ij} = (\nabla\psi_j, \nabla\psi_i)$$

When setting the confection-diffusion problem for pressure in Qh, the Neumann conditions are also applied. However, the choice of optimal conditions at the border depends on the type of border itself and the flow regime [5, 6].

In the literature, this preconditioned is called pressure con-vection-diffusion (PCD). It ensures the convergence of the method on the Krylov subspace, which depends on the calculated grid, if ν is not too small. In addition, the preconditioned is not very sensitive to the anisotropy of the grid, at least for some discretization. It can be used for both LBB-stable sampling and pressure-stabilized sampling. In addition, for a preconditioned system of equations, there are estimates of eigenvalues [12, 13].

In this paper, it is proposed to use a modified PCD preconditioned (mPCD)::

$$\hat{S}^{-1} \coloneqq \nu\widehat{M}_p^{-1} + (\alpha I + N_p)(B\widehat{M}_u^{-1}B^T)^{-1} \quad (2.8)$$

where $(N_p)_{i,j}$ = (w·∇ ψ_j,ψ_i) is a discrete advection matrix for continuous approximation of pressure; M ^ $_u$ is a diagonal approximation of the mass velocity matrix. It is known that a diagonal matrix is a good approximation in the case of regular

triangulations. However, in the case of anisotropic meshes, sometimes this approximation turns out to be weak. The matrix ($B\ M\hat{}_u^{-1} B^T$) can be considered as a mixed discretization matrix of the Poisson problem for pressure with the boundary conditions of the problem (2.9) implicitly given through the matrix M^n. This modification of the PCD preconditioned partially explains the choice of Neumann boundary conditions for the task L_p. In cases where the inertia forces can be neglected, the re-conditioner (2.8) agrees with the standard Kahu-Shabat reconditioned for the time-dependent Stokes cx problem [12].

The spacing of the eigenvalues for the pen due to the Schur complement can be limited constants c_1, C_1, independent of the grid spacing h (but possibly depending on the parameters of the task or area geometry and anisotropy):

$$0 < c_1 \le | \lambda (\hat{S}^{-1}) | \le C_1 . \qquad (2.9)$$

The proof of this fact can be found in [13] for $\alpha = 0$ and LBB-stable finite elements, for the more general case in [12].

RESULTS:

Thus, an important problem of modern medicine is the creation of effective methods of treatment and prevention of cardiovascular diseases. Mathematical modeling plays an increasing role in their development and numerical calculations of blood flow in the network of vessels with pathologies. They allow predicting surgical operations, optimizing the shape of implants, and investigating their influence on hemodynamics.

More iterations are required when the pulse wave maximum passes through the node. This is due to a decrease in accuracy when choosing an initial approximation, since the solution on the upper time layer changes more intensively. Thus, the use of this approach is limited by the value of the maximum allowable flow through the node. All the computational experiments carried out have shown that this maximum lies far beyond the boundaries of physiologically correct values. Therefore, Newton's method is computationally efficient and convenient for this task.

CONCLUSION:

Many discoveries led to the complication of sciences and specialization of scientists: only doctors began to deal with medicine, and detailed research nature of various phenomena - physicists and engineers. Over the past decades, the interaction of physics, mathematics and medicine has taken a different form. At the junction of these sciences, a new area of research arose, namely the mathematical modeling of physical processes, in our case, blood circulation.

The described approach to constructing a numerical implementation makes it possible to divide the problem into independent blocks for calculating the flow in each vessel and at each point of their docking.

Although the described quasi-one-dimensional model of global circulation provides only averaged characteristics of blood flow, it is quite convenient to use, since it does not require large computational costs generally speaking, allows real-time calculations on computers with sufficient performance. The simplicity of the model makes it possible to complicate it and thereby take into account the influence of many factors.

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