SOME REFINEMENTS OF THE LIMIT THEOREMS FOR BRANCHING RANDOM PROCESSES OF GALTON - WATSON

Zhuraev Sh. Yu.
Kokand State Pedagogical Institute

Parpiev V.
Master Student, Kokand State Pedagogical Institute

ABSTRACT
In this paper, necessary and sufficient conditions are given for the validity of the asymptotic Kolmogorov formula for the probability of continuation of a branching Galton–Watson random process.

Keywords: discrete process, generating function, formed by recursive formulas, asymptotic relation

1 Introduction

A branching random process with discrete time and one type of particles is considered [1] (ch. 2, pp. 11-52), [2] (ch.1, pp. 11-49), [3] (ch. 2, pp. 11-22).

We will say that a sequence of random variables (r.v.) $Z_0, Z_1, \dots, Z_n, \dots$ with non-negative and integer values forms a branching Galton-Watson random process (G-V) if these s. in. are defined by the following recurrence relations:

$$Z_0 = 1,\ \mathrm{Z}_n = \sum_{k=1}^{Z_{n-1}} X_k, \quad n \geq 2. \qquad (1\ .1\ )$$

Here is $X_1, X_2, \dots, X_n, \dots$ a sequence of independent c. in. non-negative and integer values with a common distribution

$$P(X_1 = n) = p_n,\ n \geq 0,\ \sum_{n=0}^{\infty} p_n = 1.$$

We assume that $P(Z_1 = X_1) = 1$the generating function c. in. Z_1

$$F(x) = Ex^{Z_1} = Ex^{X_1} = \sum_{n=0}^{\infty} p_n x^n,\ |x| \leq 1.$$

In this case, the sequences with in. $\{\mathrm{X}_n, n \geq 1\}$ and $\{\mathrm{Z}_n, n \geq 0\}$ are defined in the same probability space $(\Omega, \Im, P)$.

The following interpretation of the process G - C follows from the above: at the beginning of the process there is one particle, which $(Z_0 = 1)$, $Z_1 -$means the number of particles of the first generation. (the number of direct descendants of one particle). Therefore, $\{p_n, n \geq 0\}$ is a distribution with. in. Z_1, and the number of particles $n-$ of the th generation $Z_n\ (n \geq 2)$ is formed by recursive formulas (1 .1). Thus, generating functions (p. f.)

$$F_0(x) = Ex^{Z_0} = x,\ F_1(x) = Ex^{Z_1} = F(x),\ |x| \leq 1.$$

$$Ex^{Z_{n+1}} = F_{n+1}(x) = F_n(F(x)) = F(F_n(x)),\ n \geq 0 \ (\ 1.\ 2)$$

If discrete s. in. X takes values from the set $\{0,1,\ldots,n,\ldots\}$, then differentiating k once its a.f. $F(x)=Ex^X$ at the point $x=1$, we obtain formulas for the factorial moments $k-$ of the th order

$$m_k = F^{(k)}(1) = EX(X-1)\ldots(X-k+1) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)\,p_n$$

Factorial moments play a very significant role:

$m_1 = m = F^{'}(1) = EX$,

$b = m_2 = F^{''}(1) = EX(X-1) = EX^2 - EX$,

$m_3 = F^{'''}(1) = EX^3 - 3EX^2 + 2EX$.

Further, we will say that the process G - C $\{Z_n, n\ge 0\}$ degenerates at the moment of time n if the event occurs $\{Z_n = 0\}$. Then it is obvious that $z_{n+k}=0$ for any $k=1,2,\ldots$.

Hence $P(Z_n=0)=F_n(0)$. Probability

$$\lambda = \lim_{n\to\infty} P(Z_n=0) = \lim_{n\to\infty} F_n(0) = \lim_{n\to\infty} F(F_{n-1}(0)) = F(\lambda)$$

It follows from the last equalities that the probability of λ the process G - V degenerating is a solution to the equation $x=F(x)$. Since $x=1$ there is a trivial solution of the last equation, the probability of the degeneration of the process is determined by the equality $\lambda=\min(1,x_0)$, where x_0 satisfies the equality $x_0=F(x_0)$. It follows from the last reasoning that the probability of degeneracy λ is the smallest positive solution of the equation $F(x)=x$.

The first factorial moment $m=EZ_1$ plays an important role in the asymptotic analysis of the G–V process, being a classifying parameter for branching processes: the probability of degeneracy $\lambda=1$ at $m\le 1$, and $m>1$ the probability at $\lambda<1$. Accordingly, in the case of $m<1$, the G - C process is called subcritical, with $m=1$ critical, and in the case of $m>1$ supercritical.

As noted above, the branching random process G - C

$\{Z_n,\ n\ge 1,\ Z_0=1\}$

with one type of particles and discrete time is determined by setting the distribution of one s. in. Z_1. Really

$P(Z_0=1)=1,\ P(Z_1=k)=P(X_1=k)=p_k,\ k=0,1,2\ldots$

at $l\ne 0$

$$P(Z_{n+1}=k/Z_n=l) = P(X_1+X_2+\ldots+X_l=k) \qquad (1.3)$$

where X_i are independent and have a distribution $\{p_k, k\ge 0\}$ with a generating function $F(x)$. Besides, $P(Z_n=0/Z_{n-1}=0)=1$.

Therefore, by virtue of (1.3), the branching process Γ – B $\{Z_n, n\ge 0\}$ forms a homogeneous Markov chain with a set of states $\{0,1,\ldots,n,\ldots\}$.

Further, it is easy to see that for $m<1, x_0>1$, for $m=1, x_0=1$, for $m>1, x_0<1$. Consequently, the subcritical and critical processes Γ–B $(m\le 1)$ are degenerate with probability one, and the supercritical process $(m>1)$ is degenerate with probability $\lambda = x_0 < 1$. From what has been said, we can conclude that it has the following assertion.

If $m = EZ_1 < \infty$, then

$$\lim_{n\to\infty} P(Z_n = k) = 0, \ \forall k \geq 1,$$

$$P\left(\lim_{n\to\infty} Z_n = 0\right) = 1 - P\left(\lim_{n\to\infty} Z_n = \infty\right) = \lambda$$

The validity of the first relation of the above sentence follows from the fact that a random process $\{Z_n, n \geq 0\}$ as a Markov chain has no return states in the set $\{1, \ldots, n, \ldots\}$. The proof of the second assertion is contained in the following assertions.

Since s. in. Z_n take integer values, then degeneration is an event, which consists in the fact that $Z_n = 0$ for some $n \geq 1$.

Then $P(Z_{n+1} = 0 / Z_n = 0) = 1$, and by virtue of the continuity property of the probability measure, we have the following chain of equalities

$$P(Z_n \to 0) = 1 - P\left(\lim_{n\to\infty} Z_n = \infty\right) = P(Z_n = 0 \ \textit{для некоторого } n \geq 1) = P\left(\bigcup_{n=1}^{\infty} \{Z_n = 0\}\right) =$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} (Z_k = 0)\right) = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} F_n(0) = \lambda$$

Therefore, the probability of degeneracy of the branching process Γ–B is the smallest positive root of the equation $x = F(x)$.

2. Asymptotics of the Process Continuation Probability for Subcritical Galton–Watson Processes

.

Let $Q_n = 1 - P(Z_n = 0) = P(Z_n > 0) = P(Z_n \geq 1)$ the probability of continuation of the process G - C.

It is well known that if $m \leq 1$, then $Q_n \to 0$, for $n \to \infty$, if $m > 1$, then $\lim_{n\to\infty} Q_n = 1 - \lambda > 0$. An essential and interesting problem is the convergence of the probability asymptotics Q_n for $n \to \infty$. In 1938, A. N. Kolmogorov [4] proved that if $m < 1$ and $b = F''(1) < \infty$, then

$$Q_n = Km^n(1 + o(1)), \ n \to \infty \qquad (2.1)$$

where is K a positive constant determined by the form of the p.f. $F(\cdot)$.

Here we give a necessary and sufficient condition for the asymptotic relation (2.1) to hold.

Theorem 2.1 . If $m < 1$, then for the fulfillment of the asymptotic relation (2.1) it is necessary and sufficient that

$$\int_0^1 \frac{1 - mx - F(1-x)}{x^2} dx < \infty \qquad (2.2)$$

Proof. Let

$$F(1-x) = 1 - mx - mx\delta(x) \qquad (2.3)$$

where $\delta(x) \to 0$, at $x \to 0$. Let's put

$q_k=\sum_{j=k}^{\infty}p_j,\ G(x)=\sum_{k=0}^{\infty}q_kx^k$

Then we have

$$G(x)=\frac{1-F(x)}{1-x} \qquad (2.4)$$

Using the last equalities (2.3) and (2.4), we obtain

$$G(1-x)=\frac{1-F(1-x)}{x}=m(1+\delta(x)) \qquad (2.5)$$

It follows from (2.4) that $G(x)\to m$, for $x\to 0$. From here and from (2.3) it follows that for $0\le x\le 1$

$$\delta(x)\le 0 \qquad (2.6)$$

And, moreover, at $x\to 0$

$$\delta(x)\to 0. \qquad (2.7)$$

In fact, according to the Lagrange formula

$$F(1-x)=1-R_1(x)x=1-mx+\left(F'(1)-R_1(x)\right)x \qquad (2.8)$$

where $R_1(x)$ can be expressed as the value of the derivative $F'(x)$ at an intermediate point

$R_1(x)=F'(x\theta_x+1-\theta_x),\ 0\le\theta_x\le 1.$

It is easy to make sure that it $R_1(x)$ does not kill with $x\in[0,1]$ and besides

$0\le R_1(x)\le F'(1),\ \lim_{x\to 0}R_1(x)=F'(1)=m.$

From here and (2.8) it follows that

$$F(1-x)=1-mx+o(x),\ x\to 0 \qquad (2.9)$$

Now the proof of relations (2.6) and (2.7) follows from (2.9) in view of the equality

$\delta(x)=\frac{1-mx-F(1-x)}{x}.$

In what follows, the following assertion is used.

Lemma 2.1. fair equality

$$1-F_n(1-x)=x\prod_{k=0}^{n-1}\frac{1-F_{k+1}(1-x)}{1-F_k(1-x)} \qquad (2.10)$$

Proof.

To prove this lemma, we apply the method of mathematical induction.

When $n=1$ we have

$1-F_1(1-x)=1-F(1-x)=x\frac{1-F_1(x)}{1-F_0(1-x)}.$

Let relation (2.10) be valid for $n=p$ i.e.

$1-F_p(1-x)=x\prod_{k=0}^{p-1}\frac{1-F_{k+1}(1-x)}{1-F_k(1-x)}$

Then

$$x\prod_{k=0}^{p}\frac{1-F_{k+1}(1-x)}{1-F_k(1-x)}=x\prod_{k=0}^{p-1}\frac{1-F_{k+1}(1-x)}{1-F_k(1-x)}\cdot\frac{1-F_{p+1}(1-x)}{1-F_p(1-x)}=$$

$$=\left(1-F_p(1-x)\right)\cdot\frac{1-F_{p+1}(1-x)}{1-F_p(1-x)}=1-F_{p+1}(1-x)$$

Lemma 2.1 is proved.

From equality (2.10) it follows that

$$\frac{1-F_n(0)}{m^n}=\prod_{k=0}^{n-1}\frac{1-F_{k+1}(0)}{m\left(1-F_k(0)\right)} \qquad (2.11)$$

Let the asymptotic relation given in equality (2.1) hold.

Let us write condition (2.2) in the form

$$\int_0^1\frac{\delta(x)}{x}dx<\infty$$

The product (2.11) is transformed into equality (2.5) as follows:

$$\frac{1-F_n(0)}{m^n}=\prod_{k=0}^{n-1}\frac{1-F\left(1-\left(1-F_k(0)\right)\right)}{m\left(1-F_k(0)\right)}=\prod_{k=0}^{n-1}\left[1+\delta\left(1-F_k(0)\right)\right] \qquad (2.12)$$

Using (2.1), we obtain the equality

$$\prod_{k=0}^{n-1}\left[1+\delta\left(1-F_k(0)\right)\right]=K\left(1+o(1)\right)$$

This implies that the product (2.12) converges to a constant K i.e.

$$K=\lim_{n\to\infty}\frac{1-F_n(0)}{m^n}=\prod_{k=0}^{\infty}\left[1+\delta\left(1-F_k(0)\right)\right] \qquad (2.13)$$

Consequently, from (2.13) we obtain that

$$-\sum_{k=1}^{\infty}\delta\left(1-F_k(0)\right)<\infty.$$

By virtue of (2.1), the latter means that

$$-\sum_{n=0}^{\infty}\delta\left(Km^n\right)<\infty$$

or

$$-\int_0^{\infty}\delta\left(Km^x\right)dx<\infty.$$

Therefore, by substituting $u=Km^x$, we see that

$$\int_0^1\frac{\delta(x)}{x}dx<\infty.$$

The necessity of condition (2.2) is proved.

Remark 2.1. In the course of the proof of Theorem 2.1, an expression for the constant K in the asymptotic relation (2.1) is obtained in the form of formula (2.13).

now prove the sufficiency of condition (2.2) in Theorem 3.1. The product on the right side of equality (2.11) we set

$$I(n)=\prod_{k=0}^{n-1}\frac{1-F_{k+1}(0)}{m(1-F_k(0))}=\prod_{k=0}^{n-1}\left(1+\delta\left(1-F_k(0)\right)\right)$$

Taking into account (2.6), we can conclude that the product $I(n)$ either converges or diverges to zero. In the first case, the asymptotic relation (2.1) is satisfied.

Now let the product $I(n)$ diverge to zero i.e. $I(n)\to 0$, $n\to\infty$. In this case, the series diverges

$$-\sum_{k=0}^{\infty}\delta\left(1-F_k(0)\right)=\infty=-\sum_{k=0}^{\infty}\delta\left(m^k I(k)\right)<-\sum_{k=0}^{\infty}\delta\left(m^k\right)$$

But then the integral also diverges

$$-\int_1^{\infty}\delta\left(m^x\right)dx=\infty,$$

or

$$-\int_0^1\frac{\delta(x)}{x}dx=\infty.$$

Theorem 2.1 is proved.

Remark 2.2. 1) Note that

$$-\frac{1-mx-F(1-x)}{x^2}=\sum_{k=0}^{\infty}r_k x^k,$$

where $r_k=\sum_{i=k}^{\infty}\sum_{j=i}^{\infty}p_j,\ k0,1,2,\ldots.$

If $b=F''(1)<\infty$, then $\sum_{k=0}^{\infty}r_k=b$.

These formulas follow from the fact that the function $\frac{1-F(1-x)}{x}$ is a f.f. tails $\sum_{j=k+1}^{\infty}p_j$ of the distribution $\{p_j, j\geq 0\}$, and the function $-\frac{1-mx-F(1-x)}{x^2}$ is a p.f. distribution tails $\frac{1}{m}\sum_{k=n+1}^{\infty}\sum_{j=k+1}^{\infty}p_j$

.

2) If s. in. X takes integer non-negative values, then EX it can be calculated by the formula

$$EX=\sum_{k=1}^{\infty}P(X\geq k).$$

Indeed, in this case

$$EX=\sum_{k=0}^{\infty}kP(x=k)=\sum_{k=1}^{\infty}k\left[P(X\geq k)-P(X\geq k+1)\right]=\sum_{k=1}^{\infty}kP(x\geq k)-\sum_{k=1}^{\infty}kP(x\geq k+1)\Rightarrow$$

(Here the equality of events is used $\{X=k\}=\{X\geq k\}-\{X\geq k+1\}$).

$$\Rightarrow \sum_{k=1}^{\infty} kP(x\geq k)-\sum_{k=1}^{\infty}(k-1)P(x\geq k)=\sum_{k=1}^{\infty}P(x\geq k)$$

3) Condition (2.2) is equivalent to the existence of the expectation

$$EX\ln X=\sum_{n=1}^{\infty} n\ln nP(X=n)<\infty$$

This statement is proved by the following chain of equalities:

$$-\int_0^1 \frac{1-mx-F(1-x)}{x^2}dx=-\int_0^1 \frac{\delta(x)}{x}dx=\int_0^1\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\sum_{j=k}^{\infty}p_j\right)x^n dx=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=n}^{\infty}\sum_{j=k}^{\infty}p_j$$

Further

$$\sum_{k=n}^{\infty}\left(\sum_{j=k}^{\infty}p_j\right)=\sum_{k=n}^{\infty}\sum_{j=k}^{\infty}P(X_1=j)=\sum_{k=n}^{\infty}P(X_1\geq k).$$

Consequently

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=n}^{\infty}P(X_1\geq k)=\sum_{k=1}^{\infty}P(X_1\geq k)\sum_{k=1}^{n}\frac{1}{k+1},\ \sum_{k=1}^{n}\frac{1}{k+1}=\ln n+O(1).$$

$$-\int_0^1 \frac{1-mx-F(1-x)}{x^2}dx=\sum_{n=1}^{\infty}\ln nP(X_1\geq n)+O\left(\sum_{n=1}^{\infty}P(X_1\geq n)\right)=\sum_{n=1}^{\infty}\ln nP(X_1\geq n)+O(1)\cdot$$

Further

$$\sum_{n=1}^{\infty}\ln nP(X_1\geq n)=\sum_{n=1}^{\infty}\ln n\sum_{k=n}^{\infty}P(X_1=k)=\sum_{n=1}^{\infty}\ln n\cdot nP(X_1=n).$$

In this way

$$-\int_0^1 \frac{1-mx-F(1-x)}{x^2}dx=-\int_0^1\frac{\delta(x)}{x}dx=\sum_{n=1}^{\infty}n\ln nP(X_1=n)+O\left(\sum_{n=1}^{\infty}nP(X_1=n)\right)=$$

$$=\sum_{n=1}^{\infty}n\ln nP(X_1=n)+O(1)=\sum_{n=1}^{\infty}n\ln nP(X_1=n)+const$$

Consequently, condition (2.2) is equivalent to the existence of the integral

$$\int_0^1 \frac{1-mx-F(1-x)}{x^2}dx=\int_0^1\frac{\delta(x)}{x}dx$$

is equivalent to the condition for the existence of the mathematical expectation $EX_1\ln(1+X_1)<\infty$.

Literature

1. B.A. Sevastyanov Branching processes. Moscow. Publishing house "Science". 1971. 436 pages.
2. KBAthreya, PENey Branching Processes. Springer-Verlag. new york. Berlin. 1972, 287 p.
3. P. Haccou, P. Yagers and VAVatutin Branching processes. Cambridge university Press . 2007. 305 p .

4. A.N. Kolmogorov On the solution of a biological problem. Izv. Research Institute of Mathematics. and fur. Tomsk University. 2.2. issue 1 (1938) 1-12.
5. A.M. Yaglom Some limit theorems in the theory of branching random processes, DAN SSSR, 56.8(1947), 795-798.
6. A.V.Nagaev Refinement of some limit theorems of probability theory. Tr. ToshGU, issue 189, 55-63.
7. V.A.Vatutin Branching processes. Moscow. MIAN. 2008. 109 pp .
8. Mansurov, M., & Akbarov, U. (2021). FLATTER OF VISCOELASTIC FREE OPEROUS ROD AT THE END. Scientific Bulletin of Namangan State University, 3(3), 36-42.
9. Zhumakulov , Kh . K ., & Salimov , M . (2016). ABOUT THE METHODS OF CARRYING OUT AND THE STRUCTURE OF THE PEDAGOGICAL EXPERIMENT. Chief Editor , 80.
Esonov, M. M. (2013). Methodical techniques of a creative approach in teaching the theory of images. Vestnik KRAUNTS. Physical and Mathematical Sciences , 7 (2), 78-83.
Esonov, M. M., & Zunnunova, D. T. (2020). The development of mathematical thinking in geometry lessons through tasks for the study of image parameters. Vestnik KRAUNTS. Physical and Mathematical Sciences , 32 (3), 197-209.
12. Zharov, V. K., & Esonov, M. M. (2019). TRAINING STUDENTS OF MATHEMATICS IN SCIENTIFIC RESEARCH METHODS ON THE BASIS OF SOLVING A COMPLEX OF GEOMETRIC PROBLEMS. Continuum. Maths. Informatics. Education , (4), 10-16.
13. Esonov, M. M., & Esonov, A. M. (2016). Implementation of the methodology of creative approach in the classroom of a special course on the theory of images. Vestnik KRAUNTS. Physical and Mathematical Sciences , (1 (12)), 107-111.
Esonov, M. M. (2017). Constructing a line perpendicular to a given line. Vestnik KRAUNTS. Physical and Mathematical Sciences , (2 (18)), 111-116.
Esonov, M. M. (2016). PRACTICAL BASES OF TEACHING IMAGE METHODS TO SOLVING PROBLEMS IN THE COURSE OF GEOMETRY. In Theory and Practice of Modern Humanities and Natural Sciences (pp. 155-159).
16. Esonov, M. M. (2014). Designing the study of "Image Techniques" in the context of a creative approach to problem solving. In Theory and Practice of Modern Humanities and Natural Sciences (pp. 259-265).
17. Ergasheva , HM, Mahmudova , OY, & Ahmedova , GA (2020). GEOMETRIC SOLUTION OF ALGEBRAIC PROBLEMS. Scientific Bulletin of Namangan State University , 2 (4), 3-8.
18. Marasulova , ZA, & Rasulova , GA (2014). Information resources as a factor of integration of models and methodologies. Vestnik KRAUNC. Fiziko-Mathematicheskie Nauki , (1), 75-80.
19. Mamsliyevich, T. A. (2022). ON A NONLOCAL PROBLEM FOR THE EQUATION OF THE THIRD ORDER WITH MULTIPLE CHARACTERISTICS. INTERNATIONAL JOURNAL OF SOCIAL SCIENCE & INTERDISCIPLINARY RESEARCH ISSN: 2277-3630 Impact factor: 7.429, 11(06), 66-73.
20. Mamsliyevich, T. A. (2022). ABOUT ONE PROBLEM FOR THE EQUATION OF THE THIRD ORDER WITH A NON-LOCAL CONDITION. INTERNATIONAL JOURNAL OF SOCIAL SCIENCE & INTERDISCIPLINARY RESEARCH ISSN: 2277-3630 Impact factor: 7.429, 11(06), 74-79.
21. Muydinjanov , D. R. (2019). Holmgren problem for Helmholtz equation with the three singular coefficients. e-Journal of Analysis and Applied Mathematics , 2019 (1), 15-30.
22. Mamadaliev, B. M. (1994). Asymptotic analysis of functions of spacings.

23. Ergashev, A. A., & Tolibzhonova, Sh. A. (2020). The main components of the professional education of a teacher of mathematics. Vestnik KRAUNTS. Physical and Mathematical Sciences , 32 (3), 180-196.
24. Zunnunov, R. T., & Ergashev, A. A. (2021). Bitsadze-Samarsky type problem for mixed type equation of the second kind in a domain whose elliptic part is a quarter of the plane. In Fundamental and applied problems of mathematics and computer science (pp. 117-20).
25. Zunnunov, R. T., & Ergashev, A. A. (2016). A problem with a shift for a mixed-type equation of the second kind in an unbounded domain. Vestnik KRAUNTS. Physical and Mathematical Sciences , (1 (12)), 26-31.
26. Zunnunov, R. T., & Ergashev, A. A. (2017). Boundary value problem with a shift for a mixed type equation in an unbounded domain. In Actual problems of applied mathematics and physics (pp. 92-93).
27. Zunnunov, R. T., & Ergashev, A. A. (2016). A problem with a shift for a mixed-type equation of the second kind in an unbounded domain. Vestnik KRAUNTS. Physical and Mathematical Sciences , (1 (12)), 26-31.
28. Zunnunov , R.T., & Ergashev, A.A. (2016). PROBLEM WITH A SHIFT FOR A MIXED-TYPE EQUATION OF THE SECOND KIND IN AN UNBOUNDED DOMAIN. Bulletin KRASEC. Physical and Mathematical Sciences , 12 (1), 21-26.
29. Ergashev, A. A., & Talibzhanova, Sh. A. (2015). Technique for solving the Bitsadze-Samarsky problem for an elliptic type equation in a half-strip. In Theory and Practice of Modern Humanities and Natural Sciences (pp. 160-162).
Alyaviya, O., Yakovenko, V., Ergasheva, D., Usmanova, Sh., & Zunnunov, H. (2014). Evaluation of the intensity and structure of dental caries in students with normal and reduced function of the salivary glands. Stomatologiya , 1 (3-4 (57-58)), 34-38.
30. Marasulova, Z. A., & Rasulova, G. A. (2014). Information resource as a factor of integration of models and methods. Vestnik KRAUNTS. Physical and Mathematical Sciences , (1(8)), 75-80.
31. Rasulova, G. A., Ahmedova, Z. S., & Normatov, M. (2016). THE METHOD OF STUDYING MATHEMATICAL TERMS IN ENGLISH IN THE PROCESS OF LEARNING. Scientist of the 21st Century , 65.
32. Rasulova, G. A., Ahmedova, Z. S., & Normatov, M. (2016). EDUCATION ISSUES LEARN ENGLISH LANGUAGE IN TERMS OF PROCESSES. The 21st Century Scientist , (6-2(19)), 62-65.
33. Rasulova , G. (2022). CASE STADE AND TECHNOLOGY OF USING NONSTANDARD TESTS IN TEACHING GEOMETRY MODULE. Eurasian journal of Mathematical theory and computer sciences , 2 (5), 40-43.
34. Ergasheva, H. M., Mahmudova, O. Y., & Ahmedova, G. A. (2020). GEOMETRIC SOLUTION OF ALGEBRAIC PROBLEMS. Scientific Bulletin of Namangan State University, 2(4), 3-8.
35. Muydinjonov, Z., & Muydinjonov, D. (2022). INFORMATION, COMMUNICATION AND TECHNOLOGY (ICT) IS FOR TEACHER AND STUDENT.
36. Muydinjonov, Z., & Muydinjonov, D. (2022). VIRTUAL LABORATORIES. Eurasian Journal of Academic Research, 2(6), 1031-1034.
37. Muydinjanov, D. R. (2019). Holmgren problem for Helmholtz equation with the three singular coefficients. e-Journal of Analysis and Applied Mathematics, 2019(1), 15-30.
38. Rahmatullaev, M. M., Rafikov, F. K., & Azamov, S. (2021). On the Constructive Description of Gibbs Measures for the Potts Model on a Cayley Tree. Ukrainian Mathematical Journal, 73(7), 1092-1106.

39. Rahmatullaev , M., Rafikov , F.K. , & Azamov , SK (2021). On constructive descriptions of Gibbs measures for the Potts model on the Cayley tree. Ukrains' kyi Matematychnyi Zhurnal , 73 (7), 938-950.
40. Petrosyan , VA, & Rafikov , FM (1980). Polarographic study of aliphatic nitro compounds. Bulletin of the Academy of Sciences of the USSR, Division of chemical science , 29 (9), 1429-1431.
41. Formanov, S. K., & Jurayev, S. (2021). On Transient Phenomena in Branching Random Processes with Discrete Time. Lobachevskii Journal of Mathematics, 42(12), 2777-2784.