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SOME REFINEMENTS OF THE LIMIT THEOREMS FOR GALTON - WATSON BRANCHING RANDOM PROCESSES

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ANNOTATION
The necessary and sufficient conditions for the validity of the asymptotic Kolmogorov formula for the probability of continuation of a Galton – Watson branching random process are presented. It is proved that these conditions can be expressed in moment characteristics of the form $E|X|\ln(1+|X|)$.

Keywords: interpretation, A consistent application of Lemma, expectation, independent in the set

1. Introduction

A branching random process with discrete time and one type of particles is considered [1] (Ch. 2, pp. 11-52), [2] (Ch. I, pp. 11-49), [3] (Ch. 2, pp. 11-22).

Let us say that a sequence of random variables (r.v.) $Z_0, Z_1, \ldots, Z_n, \ldots$ with nonnegative and integer values forms a Galton-Watson branching random process (G-W) if these r.v. are determined by the following recurrence relations:

$$Z_0 = 1,\ Z_n = \sum_{k=1}^{Z_{n-1}} X_k,\quad n \geq 2. \qquad (1.1)$$

Here $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent r.v. with non-negative and integer values with a common distribution

$$P\left(X_1 = n\right) = p_n,\ n \geq 0,\ \sum_{n=0}^{\infty} p_n = 1.$$

We will assume that $P\left(Z_1 = X_1\right) = 1$ and the generating function of r.v. Z_1 are:

$$F\left(x\right) = Ex^{Z_1} = Ex^{X_1} = \sum_{n=0}^{\infty} p_n x^n,\ |x| \leq 1.$$

The sequences of r.v. $\{X_n, n \geq 1\}$ and $\{Z_n, n \geq 0\}$ are defined in the same probability space $\left(\Omega, \Im, P\right)$.

The following interpretation of the G-W process results from the above: at the beginning of the process there is one particle $\left(Z_0 = 1\right)$, $Z_1 -$ means the number of particles of the first generation (the number of direct descendants of one particle). Therefore, $\left\{p_n, n \geq 0\right\}$ is the distribution of r.v., and the number of particles of the *n*-th generation $Z_n\left(n \geq 2\right)$ is formed by recurrent formulas (1.1). Thus, the generating functions (g.f.) are:

$$F_0\left(x\right) = Ex^{Z_0} = x,\ F_1\left(x\right) = Ex^{Z_1} = F\left(x\right),\ |x| \leq 1.$$

To find g.f. of the number of particles of the *n*-th generation

$F_n(x) = Ex^{Z_n}$, $n = 2,3,\ldots$

we will use the following simple assertion.

Lemma 1.1. Let $\nu = \nu(\omega)$ be r.v. with values from the extended set of natural numbers $N_0 = \{0,1,\ldots,n,\ldots\}$ and S_ν be a random sum (i.e., the sum of a random number of r.v.)

$S_\nu = X_1 + \ldots + X_\nu$.

If the sequence of r.v.

$\nu, X_1, X_2, \ldots, X_n, \ldots$

is independent in the set, in particular, ν does not depend on any r.v. $X_1,\ldots,X_n,\ldots$ then g.f. is

$$Ex^{S_\nu} = G(F(x)), \quad |x| \le 1 \tag{1.2}$$

Where g.f. is $G(x) = Ex^\nu = \sum_{n=0}^{\infty} P(\nu = n)x^n$.

Proof. In what follows, the "narrowed" mathematical expectation r.v. $X = X(\omega)$ is used; it is defined by the following formula

$E(X;B) = \int_B XdP = \int_B X(\omega)\,\mathrm{P}(\mathrm{d}\,\omega),\ B \in \Im$.

Since the system of events $\{\nu = n\}$, $n = 0,1,\ldots$ forms a complete group, by the formula for the total mathematical expectation we have

$$Ex^{S_\nu} = \sum_{n=0}^{\infty} E\left(x^{S_\nu};\nu = n\right) = \sum_{n=0}^{\infty} E\left(x^{S_n};\nu = n\right) = \sum_{n=0}^{\infty} Ex^{S_n} \cdot P(\nu = n) =$$

$$= \sum_{n=0}^{\infty} P(\nu = n) F^n(x) = G(F(x)),\ |x| \le 1.$$

Therefore, the above Lemma 1.1 is proved.

A consistent application of Lemma 1.1 allows us to verify the validity of the following recurrence formulas:

$$Ex^{Z_{n+1}} = F_{n+1}(x) = F_n(F(x)) = F(F_n(x)),\ n \ge 0 \tag{1.3}$$

Indeed, from (1.1) by Lemma 1.1 it follows that

$F_0(x) = x,\ F_1(x) = F(x),\ F_2(x) = F_1(F(x)) = F(F(x))$

and so on. A rigorous proof of the recurrence relation (1.3) consists in the application of the method of mathematical induction.

If a discrete r.v. takes values from the set $\{0,1,\ldots,n,\ldots\}$, then differentiating its g.f. $F(x) = Ex^X$ at the point $x = 1$ for k times, we obtain formulas for the factorial moments of the k-th order

$$m_k = F^{(k)}(1) = EX(X-1)\ldots(X-k+1) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)\,p_n$$

Factorial moments play a very important role:

$m_1 = m = F'(1) = EX$,

$b = m_2 = F''(1) = EX(X-1) = EX^2 - EX$,

$m_3 = F'''(1) = EX^3 - 3EX^2 + 2EX$.

In what follows, we assume that the G-W process $\{Z_n, n \ge 0\}$ degenerates at the n point of time if an event $\{Z_n = 0\}$ occurs. Then it is obvious that $z_{n+k} = 0$ for any $k = 1, 2, \ldots$.

Hence $P(Z_n = 0) = F_n(0)$. The probability is

$$\lambda = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} F_n(0) = \lim_{n\to\infty} F(F_{n-1}(0)) = F(\lambda)$$

From the last equalities it follows that the probability of degeneration λ of the G-W process is a solution to the equation $x = F(x)$. Since $x = 1$ is a trivial solution to the last equation, the probability of the process degeneration is determined by equation $\lambda = \min(1, x_0)$, where x_0 satisfies $x_0 = F(x_0)$. From the last considerations it follows that the probability of degeneration λ is the least positive solution to the equation $F(x) = x$.

The first factorial moment $m = EZ_1$ plays an important role in the asymptotic analysis of the G-W process, being a classifying parameter for branching processes: the probability of degeneration is $\lambda = 1$ at $m \le 1$, and $\lambda < 1$ at $m > 1$. Accordingly, in the case of $m < 1$, the G-W process is called subcritical, at $m = 1$
it is critical, and at $m > 1$ it is supercritical.

As noted above, the G-W branching random process
$\{Z_n,\ n \ge 1,\ Z_0 = 1\}$
with one type of particles and discrete time is determined by setting the distribution of one r.v. Z_1. Indeed,
$P(Z_0 = 1) = 1,\ P(Z_1 = k) = P(X_1 = k) = p_k,\ k = 0, 1, 2\ldots$
at $l \ne 0$

$$P(Z_{n+1} = k / Z_n = l) = P(X_1 + X_2 + \ldots + X_l = k) \qquad (1.4)$$

X_i are independent and have a distribution $\{p_k, k \ge 0\}$ with a generating function $F(x)$. Besides,
$P(Z_n = 0 / Z_{n-1} = 0) = 1.$

Consequently, by virtue of (1.4), the G-W branching process $\{Z_n, n \ge 0\}$ forms a homogeneous Markov chain with a set of states $\{0, 1, \ldots, n, \ldots\}$.

Then, it is easy to see that at $m < 1, x_0 > 1$, at $m = 1, x_0 = 1$, at $m > 1, x_0 < 1$. Consequently, the subcritical and critical G-W processes $(m \le 1)$ are degenerating processes with probability one, and the supercritical process $(m > 1)$ degenerates with probability $\lambda = x_0 < 1$. From above, we can conclude that the following assertion holds.

If $m = EZ_1 < \infty$, then

$$\lim_{n\to\infty} P(Z_n = k) = 0,\ \forall k \ge 1,$$

$$P\left(\lim_{n\to\infty} Z_n = 0\right) = 1 - P\left(\lim_{n\to\infty} Z_n = \infty\right) = \lambda$$

The validity of the first relation in the above assertion follows from the fact that a random process $\{Z_n, n \ge 0\}$ as a Markov chain does not have recurrent states in the set $\{1,\ldots,n,\ldots\}$. The proof of the second assertion is in the following statements.

Since r.v. Z_n take integer values, then degeneration is an event in which $Z_n = 0$ for some $n \ge 1$.

Then, $P(Z_{n+1} = 0 / Z_n = 0) = 1$ and by virtue of the continuity property of the probability measure, we have the following chain of equalities *– for some*

$$P(Z_n \to 0) = 1 - P\left(\lim_{n\to\infty} Z_n = \infty\right) = P(Z_n = 0 \ \text{для некоторого } n \ge 1) = P\left(\bigcup_{n=1}^{\infty}\{Z_n = 0\}\right) =$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n}(Z_k = 0)\right) = \lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} F_n(0) = \lambda$$

Hence the probability of degeneration of the G-W branching process is the least positive root of the equation $x = F(x)$.

2. Asymptotics of the probability of the process continuation for subcritical Galton-Watson processes.

Let $Q_n = 1 - P(Z_n = 0) = P(Z_n > 0) = P(Z_n \ge 1)$ be the probability of G-W process continuation.

It is well known that if $m \le 1$, then $Q_n \to 0$, at $n \to \infty$, if $m > 1$, then $\lim_{n\to\infty} Q_n = 1 - \lambda > 0$. An essential and interesting problem is to find an asymptotic behavior of the probability Q_n as $n \to \infty$. In 1938, A. N. Kolmogorov [4] proved that if $m < 1$ and $b = F''(1) < \infty$, then

$$Q_n = Km^n(1 + o(1)),\ n \to \infty \qquad (2.1)$$

where K is a positive constant determined by the form of the g.f. $F(\cdot)$

Here we give a necessary and sufficient condition for the realization of asymptotic relation (2.1).

Theorem 2.1. If $m < 1$, then for the asymptotic relation (2.1) to hold, it is necessary and sufficient that

$$\int_0^1 \frac{1 - mx - F(1-x)}{x^2} dx < \infty \qquad (2.2)$$

As follows from Theorem 2.1, the convergence (existence) of integral (2.2) ensures the validity of the asymptotic representation (2.1) for the probability of continuation of subcritical G-W branching processes. In turn, the convergence of integral (2.2) is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} n \ln n P(X = n) \qquad (3.1)$$

and this series is often seen in limit theorems for noncritical G-W branching processes ([7], 5.1-5.4, pp. 29-42).

In this section, we present the following theorem on the existence of integral (2.2).

Theorem 2.2. Let $F(x)=\sum_{n=0}^{\infty} p_n x^n$ be the probabilistic generating function and let

$$0<m=F'(1)<\infty .$$

Then the integral

$$\int_0^1 \frac{1-mx-F(1-x)}{x^2}dx$$

exists if and only if

$$\int_0^1 u^{-2}\left(F\left(e^{-\frac{u}{m}}\right)-e^{-u}\right)du<\infty \qquad (3.2)$$

Proof. Let X be a nonnegative integer random variable with g.f. $F(x)$.The latter means that

$$F\left(e^{-\frac{u}{m}}\right)=E\left(e^{-\frac{u}{m}X}\right)$$

Therefore, we have the representation

$$F\left(e^{-\frac{u}{m}}\right)-e^{-u}=E\left(e^{-\frac{u}{m}X}\right)-e^{-u}=E\left(e^{-\frac{u}{m}X}-1+\frac{uX}{m}\right)+1-u-e^{-u}$$

It is obvious that

$$\int_0^1 \frac{e^{-u}-1+u}{u^2}du=O(1) \qquad (3.3)$$

The latter implies that the convergence of integral (3.2) is equivalent to

$$\int_0^1 \left(u^{-2}E\left(e^{-\frac{uX}{m}}\right)-1+\frac{uX}{m}\right)du<\infty \qquad (3.4)$$

Let $G(u)=P(X<mu)$. Then the left-hand side of (3.4) is equal to

$$\int_0^{\infty}\left(\int_0^1\left(\frac{e^{ux}-1+ux}{u^2}\right)du\right)dG(x) \qquad (3.5)$$

But

$$\int_0^1 u^{-2}\left(e^{-ux}-1+ux\right)du=x\int_0^x u^{-2}\left(e^{-u}-1+u\right)du \qquad (3.6)$$

After integrating by parts the right-hand side of the last equation (3.6) twice, it is easy to see that for large values of x

$$x\int_0^x u^{-2}\left(e^{-u}-1+u\right)du=x\ln x+O(1) \qquad (3.7)$$

Therefore, from (3.6) and (3.7) we obtain

$$\lim_{x\to\infty}\left(\int_0^1 \frac{e^{-ux}-1+ux}{u^2}du\right)(x\ln x)^{-1}=1 \qquad (3.8)$$

Now from the last relation (3.8) it follows that the integral (3.5) and the integral $\int_0^\infty x\ln x dG(x)$ converge simultaneously.

Since r.v. X is discrete, the following equation holds:

$$\int_0^\infty x\ln x dG(x) = m\sum_{n=1}^\infty n\ln n p_n$$

It follows from the last equation that the convergence of integral (3.2) is equivalent to the convergence of the series $\sum_{n=1}^\infty (n\ln n)\, p_n$.

The proof of Theorem 2.2 follows from Theorem 2.1.

It follows from equation that

$$-m\delta(x) = m\tilde{\delta}(x) = m - \frac{1-F(1-x)}{x},\ 0 \le x \le 1$$

and conditions (2.2) can be written in the form

$$\int_0^1 \frac{\tilde{\delta}(x)}{x} dx < \infty \qquad (3.9)$$

It should be noted that the function $\tilde{\delta}(x)$ at zero $(x=0)$ is determined by continuity, i.e., $\tilde{\delta}(0)=0$.

Theorem 2.3. Let the conditions of Theorem 2.2 be satisfied. Then

a) the function $\tilde{\delta}(x)$ is non-negative and non-decreasing.

b) in order for all $c>0$ and $0<\rho<1$ the series $\sum_{n=1}^\infty \tilde{\delta}(c\rho^n)$ be convergent, it is necessary and sufficient to satisfy condition (3.9).

Proof. Since $\lim_{x\to 0+}\tilde{\delta}(x)=0$ and $F(x)$ is a convex function, we have $\tilde{\delta}(x)\ge 0$ and $\tilde{\delta}'(x)\ge 0$. Consequently, the part of the theorem holds. Part b) of Theorem 2.3 is proved in the following reasoning.

Note, that

$$\tilde{\delta}(x) = \sum_{j=1}^\infty P_j\left[\sum_{k=0}^{j-1}\left(1-(1-x)^k\right)\right]$$

From this representation it follows that the function $\tilde{\delta}(x)$ is a monotonically decreasing function and therefore the equality relation holds:

$$\left\{\sum_{n=1}^\infty \tilde{\delta}(c\rho^n) < \infty\right\} \Leftrightarrow \left\{\int_0^\infty \tilde{\delta}(c\rho^n) du < \infty\right\} \qquad (3.10)$$

The right-hand side of this relation holds if and only if

$$\int_0^1 v^{-1}\tilde{\delta}(v) dv < \infty \qquad (3.11)$$

The latter can be verified by changing the variable $v = \rho^u$. Further

$$v^{-1}\tilde{\delta}(v) = \left[F(1-v) - 1 + mv\right]v^{-2} =$$

$$= \left[F\left(e^{-\frac{u}{m}}\right) - e^{-u} + m\left(1 - e^{-\frac{u}{m}} - \frac{u}{m}\right) + \left(e^{-u} - 1 + u\right)\right]u^2\left(uv^{-1}\right)^2 \qquad (3.12)$$

where $1 - v = e^{-\frac{u}{m}}$.

With simple reasoning, one can verify that the following relations hold for any $0 < a \le b < \infty$

$$1 \le \sup_{a \le x \le b} \frac{x}{1 - e^x} \le A < \infty \qquad (3.13)$$

$$\sup_{x \ge 0}\left|e^{-x} - 1 + x\right| \le \frac{1}{2} \qquad (3.14)$$

From (3.12) - (3.14) it follows that for some $a > 0$

$$\left\{\int_0^1 v^{-1}\tilde{\delta}(v) < \infty\right\} \Leftrightarrow \left\{\int_0^a \left(F\left(e^{-\frac{u}{m}}\right) - e^{-u}\right)u^{-2}du < \infty\right\} \qquad (3.15)$$

The final proof of part b) of Theorem 2.3 follows from Theorem 2.2 in view of relations (3.9) - (3.11)

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