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SOME BASICS OF BESSEL GENERATING AND RELATED FUNCTIONS THE STUDY OF SINGLE AND DOUBLE INTEGRATION FORMULA

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Abstract
The Bessel functions fall into two categories, those with even symmetry in x for even orders n and those with odd symmetry in x for odd orders n. Solving Laplace's equation and the Helmholtz equation separately in cylindrical or spherical dimensions leads to Bessel's equation. Thus, Bessel functions play a crucial role in many issues involving wave propagation and static potentials. Bessel functions may be understood in a unique way via the lens of algebraic geometry. Many families of generating functions including products of Bessel and related functions are derived using a unique methodology developed by the authors, which relies on the combination of operational techniques with certain particular multivariable and multi-index polynomials. We determine the generating function and use it to establish a number of important classical conclusions and recurrence connections. To further understand the Bessel functions at certain values, we employ these recurrence relations. We also demonstrate a different way to derive the first Bessel function from the generating function.

Keywords: Bessel Functions, Fractional, Derivative, Integral, Multivariable.

INTRODUCTION
Canonical solutions y(x) of Bessel's differential equation are called Bessel functions, a term coined by Daniel Bernoulli and then extended by Friedrich Bessel.

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\alpha^2\right)y=0$$

order of the Bessel function for a complex number α. It is customary to define separate Bessel functions for α and -α, such that the Bessel functions are mainly smooth functions of α, even if they both result in the same differential equation. When is an integer or a half-integer, they are the most relevant examples. As these integer α Bessel functions exist in the cylindrical coordinate solution of Laplace's equation, they are sometimes referred to as the cylinder functions or the cylindrical harmonics. Solving the Helmholtz equation in spherical coordinates yields spherical Bessel functions with a half-integer α.
During the last three decades, there has been a proliferation of research into the theoretical foundations and practical applications of fractional calculus throughout the physical sciences, mathematics, and engineering. Different phenomena in physics, biology, chemistry, engineering, diffusive systems, electrical systems, control theory, polymer, biophysics, thermodynamics, viscoelasticity, electroanalytical chemistry, etc. can be successfully formulated using fractional derivatives and integrals, which are nonlocal operators despite being of integer order. Since it is not always feasible to discover analytical solutions to fractional differential equations, numerical approaches for these

problems have undergone substantial development in recent years, in keeping with the growing relevance of the fractional calculus and its applications. Based on the authors' current understanding, all numerical approaches in this domain may be roughly classified as either local operator-based numerical methods or global operator-based numerical methods. In contrast to the global nature of the Galerkin, Petrov-Galerkin, and pseudospectral (also referred to as the collocation) techniques, the well-known finite difference and finite element approaches are fundamentally based on the local operators. Both the global and local numerical approaches have their benefits and drawbacks.

LITERATURE REVIEW

Vynnyts'kyi, B. & Khats, Ruslan & Sheparovych, Iryna (2020) In the set $L2$, we discover a criteria of unconditional basicity for the system $(\sqrt{x\rho_k}J_{\nu}(x\rho_k):k\in\mathbb N)$ where $(rho k:kinmathbb N)$ is a series of unique nonzero complex integers, and $J nu$ is the Bessel function of the first type with index $nugeq-1/2$$\nu\geq-1/2$ and $(\rho_k:k\in\mathbb N)$.

Kalf, Hubert & Okaji, Takashi & Yamada, Osanobu (2019) Bessel functions with an integrable weight that may be singular at the origin are considered, and explicit and partially sharp estimates of their squared integrals are provided. They are independent of the radius of the sphere and uniform with regard to the order of the Bessel functions, while also providing explicit constraints for some smoothing estimations. It is shown that for more specialized weights, these constraints are Hölder continuous, and that a Hölder constant is also dependent in this way. An example of how these findings might be put to use is shown by deriving a uniform resolvent estimate of the free Dirac operator with mass in dimensions.

Yakubovich, Semyon (2020) Squares of Bessel functions of the first and second order $J nu(z), Ynu(z)$ are presented and studied as discrete analogues of the index transformations. The relevant inversion theorems are proved for appropriate families of functions and sequences.

Zayed, Hanaa & Bulboacă, Teodor (2022) Some of the geometric features of the normalization of the extended Bessel functions Uσ,r (σ, r ∈ C) defined by Uσ,r (z) = z + ∞ j=1 (−r)j 4j(1)j(σ)j zj+1 have already been reported. Using a novel approach, this research aims to round out previous findings in the literature. To ensure that U,r is starlike or convex of order α (0 ≤ α ≤ 1) on the open unit disk, we first utilized an identity for the logarithmic of the gamma function and an inequality for the digamma function to set adequate requirements on the parameters. In addition, the starlikeness and convexity of U,r have been taken into account, with the starlikeness of the power series f f (z) = ∞ j=1 Ajzj serving as the central notion of the proofs and the traditional Alexander theorem serving as a bridge between the classes of starlike and convex functions. To demonstrate that our requirements do not conflict, we provided a brief proof. Finally, the values of (z cos√z) ∗ Uσ,r and (sin z) ∗ Uσ,r (z2) z, where "∗" denotes the convolution between the power series, have been shown to be very near to convex.

BASICS OF BESSEL FUNCTIONS

Their Characteristics we will now verify some of the essential features of the two Bessel functions that we have generated.

The Generating Function

The Bessel functions' generating function may be used to verify a number of their characteristics. In this first part, we will discuss generating functions and demonstrate the existence of a generating function for the Bessel functions.

Definition 1: Power series are infinite sequences of the

$$\sum_{i=0}^{\infty} a_i z^i \quad (1)$$

In which the ai's are values determined by some metric or rule.

Definition 2: Specifically, the generating function of a given function a_n is the function $G(a_n; x)$ whose power series contains a_n as the coefficient of x^n.

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n \quad (3)$$

Proposition 1. We now possess

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n \quad (4)$$

That is to say, the role

$$e^{\frac{x}{2}(z-z^{-1})}$$

the primary Bessel function's generating function.

(This is a Laurent series, not a power series, since it contains negative degree components. This is going to be significant.)

Proof. Remember the symbol for the power series

$$e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!}.$$

We now possess

$$e^{\frac{x}{2}z - \frac{x}{2}\frac{1}{z}} = e^{\frac{x}{2}z} e^{-\frac{x}{2}\frac{1}{z}}$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{x}{2}z)^m}{m!} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2z})^k}{k!}$$

$$= \sum_{m=0}^{\infty} \frac{(x/2)^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} z^{-k}$$

Keep in mind that the Cauchy Product is the expression for the multiplication of two infinite power series, i.e., if you have two power series,

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$

whose products may likewise be written as power series in the disc |z| < R if and only if each has a radius of convergence R > 0:

$$(fg)(z) = \sum_{n=0}^{\infty} c_n z^n$$

Where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

With this characteristic in mind, we get:

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{(x/2)^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} z^{-k} &= \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{m-k=n \\ m,k\geq 0}} \frac{(-1)^k (\frac{x}{2})^{m+k}}{m!k!} \right) z^{m-k} \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} (\frac{x}{2})^{n+k+k} \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} (\frac{x}{2})^{2k} (\frac{x}{2})^n \right) z^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x) z^n.\end{aligned}$$

SOME FAMILIES OF GENERATING FUNCTIONS FOR THE BESSEL AND RELATED FUNCTIONS

Dattoli et al. generated generating functions of the kind by applying specific operational principles.

$$S_{\{p\}}(\{x\};t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} J_{n+p_1}(x_1) \cdots J_{n+p_m}(x_m) \tag{5}$$

({x} = x1, . . . , xm; {p} = p1, . . . , pm),

In which the Bessel function $J_n(x)$ is a spherical one. In the right-hand side of equation (5), the indices p1,..., pm are not required to be integers. Functions like (5) are derived using an approach that combines operational techniques with certain families of special functions using many indices and many variables. It may be generalized to the case of Bessel functions on a sphere. The primary goal of this study is to propose an expansion of the aforementioned method and to demonstrate how this expanded approach leads to additional generalizations, such as hybrid generating functions. To introduce Proposition 2, we first show how a well-known generating function may be derived.

Proposition 2. The following relation holds for the generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} J_{n+\nu}(x) = \left(\frac{x}{x-2t} \right)^{\nu/2} J_\nu\left(\sqrt{x^2-2xt}\right) \quad (\nu \in \mathbb{C}) \tag{6}$$

Proof. In the well-known derivative formula, if we multiply both sides by

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(x^{-\nu}J_\nu(x)\right)=\frac{(-1)^n}{x^{n+\nu}}\,J_{n+\nu}(x)$$
$$(\nu\in\mathbb{C};\quad n\in\mathbb{N}_0:=\{0,1,2,\dots\}) \qquad (7)$$

by $\tau^n/n!$ and adding each side from n = 0 to n =∞, we get (7), which tells us

$$\exp\left(\frac{\tau}{x}\frac{d}{dx}\right)\left(x^{-\nu}\,J_\nu(x)\right)=x^{-\nu}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\tau}{x}\right)^n\,J_{n+\nu}(x)\quad(\nu\in\mathbb{C}) \qquad (8)$$

Use of the Process Identity

$$\exp\left(\frac{\tau}{x}\frac{d}{dx}\right)f(x)=f\left(\sqrt{x^2+2\tau}\right) \qquad (9)$$

After plugging τ = −xt into (8), we get (6) without much effort. The so-called Tricomi-Bessel function, introduced by, provides a different approach.

$$C_n(x):=\sum_{r=0}^{\infty}\frac{(-1)^r\;x^r}{r!\,(n+r)!}, \qquad (10)$$

This is connected to the Bessel function $J_n(x)$ on a cylinder by

$$C_n(x)=x^{-n/2}\,J_n\left(2\sqrt{x}\right)\quad\text{and}\quad J_n(x)=\left(\frac{x}{2}\right)^n\,C_n\left(\frac{x^2}{4}\right) \qquad (11)$$

using the function of generation

$$\sum_{n=-\infty}^{\infty}t^n\,C_n(x)=\exp\left(t-\frac{x}{t}\right)$$

From (10)'s definition, it's easy to deduce that

$$\left(\frac{d}{dx}\right)^n\,C_l(x)=(-1)^n\;C_{n+l}(x) \qquad (12)$$

To ensure

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\,C_{n+l}(x)=\exp\left(-t\,\frac{d}{dx}\right)C_l(x)=C_l(x-t)$$

Hence, considering (11), produces the generation function (6). This clearly concludes our non-identity-based (operational) (9) alternative derivation of the generating function (6).

A Class of Multivariable Bessel Functions

Certain families of multivariate Bessel functions play a significant role in both pure and practical mathematics. For instance, the generating function defines a Bessel function with two variables and one argument.

$$\sum_{n=-\infty}^{\infty}t^n\,J_n(x,y;\tau)=\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2\tau-\frac{1}{t^2\tau}\right)\right]$$

as shown by the show itself:

$$J_n(x,y;\tau)=\sum_{l=-\infty}^{\infty}\tau^l\,J_{n-2l}(x)\,J_l(y)$$

For our needs, we may introduce the one-parameter, two-variable version of the Tricomi-Bessel function by assuming that it meets the following identities:

$$\sum_{n=-\infty}^{\infty}t^n\,C_n(x,y;\tau)=\exp\left(t-\frac{x}{t}+t^2\tau-\frac{y}{t^2\tau}\right) \tag{13}$$

$$C_n(x,y;\tau)=\sum_{l=-\infty}^{\infty}\tau^l\,C_{n-2l}(x)\,C_l(y)\,,$$

$$J_n(x,y;\tau)=\left(\frac{x}{2}\right)^n\,C_n\left(\frac{x^2}{4},\frac{y^2}{4};\frac{2y}{x^2}\,\tau\right)$$

And

$$C_n(x,y;\tau)=x^{-n/2}\,J_n\left(2\sqrt{x},2\sqrt{y};\frac{x}{\sqrt{y}}\,\tau\right)$$

CONCLUSION

Bessel functions are widely used in mathematics and physics, and their study is crucial to the field of wave mechanics. A further extension of the sine function, the Bessel function. Vibrations in a media with varying characteristics, membrane vibrations in a disc, string vibrations under varying tension, etc., are all possible explanations. Bessel functions (BFs) are ubiquitous in mathematical physics and are known to be both fascinating and useful. Daniel Bernoulli's work on the oscillations of a hanging chain, Euler's hypothesis of the vibration of a circular membrane, and Bessel's investigations into planetary motion all contributed to the development of the BFs. Several issues in potential theory and diffusion that involve cylindrical symmetry have recently been shown to benefit from the use of BFs in the physical and engineering sciences.

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