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ROTA-BAXTER OPERATORS ON DIHEDRAL QUANDLES

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ABSTRACT
Rota-Baxter operators on racks and quandles were introduced by Bardakov and Bovdi. The main goal of this paper is to study Rota-Baxter operators on dihedral quandles.
Let $R_2 = \{a_0, \ a_1\}$ be a 2-element dihedral quandle. Any mapping

$$B: R_2 \to R_2$$

is a Rota-Baxter operator on R_2. This raises the following questions:

Question. What can be said about Rota-Baxter operators on an arbitrary dihedral quandle R_n ?

Keywords: Rota-Baxter operators, dihedral quandle, binary operation, automorphisms

Introduction
Rota--Baxter operators for commutative algebras first appear in the paper of G. Baxter [7].
For basic results and the main properties of Rota--Baxter algebras see [10]
In [10] was define the Rota--Baxter operator on groups.
A group G with a Rota--Baxter operator B is called a Rota--Baxter group. In the paper [11] it was proved, that if (G, B) is a Rota--Baxter Lie group, then the tangent map of B at the identity is a Rota-Baxter operator of weight 1 on the Lie algebra of the Lie group G
In [9] was proved that Rota-Baxter operators on G as on a Hopf algebra are in one-two-one correspondence with Rota-Baxter operators of weight 1 on G.

2.1. **Quandles.** A quandle is a non-empty set Q with a binary operation $(x, y) \to (x * y)$ satisfying the following axioms:
(Q1) $x * x = x$ for all $x \in Q$;
(Q2) for any $x, y \epsilon Q$ there exists a unique $z \in Q$ such that $x = z * y$;
(Q3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$
An algebraic system satisfying only (Q2) and (Q3) is called a k . Many interesting examples of quandles come from groups.

- G is a group, then the binary operation $a * b = b^{-1}ab$ turns G into the quandle $\mathrm{Conj}(G)$ called the Conjugation quandl of G.
- A group G with the binary operation $a * b = ba^{-1}b$ turns the set G into the quandle $\mathrm{Core}(G)$ called the core quandle of G. In particular, if $G = Z_n$ the cyclic group of order n then it is called the dihedral quandle and denoted by R_n.
- Let G be a group and $\varphi \in \mathrm{Aut}(G)$ Then the set G with binary operation

$a * b = \varphi(ab^{-1})b$ forms a quandle $\mathrm{Alex}(G, \ \varphi)$ referred as the generalized Alexander quandle of G with respect to φ.

A quandle Q is called trivial if $x * y = x$ for all $x, y \in Q$. Unlike groups, a trivial quandle can have arbitrary number of elements. We denote the n element trivial quandle by T_n and an arbitrary trivial quandle by T. Notice that the axioms $(Q2)$ and $(Q3)$ are equivalent to the map $S_x: Q \to Q$ given by

$$S_x(y) = y * x$$

being an automorphism of Q for each $x \in Q$. These automorphisms are called inner automorphisms, and the group generated by all such automorphisms is denoted by $\mathrm{inn}(X)$. A quandle is said to be connected if it admits a transitive action by its group of inner automorphisms. For example, dihedral quandles of odd order are connected, whereas that of even order are disconnected. A quandle X is called involutary if $S_x^2 = \mathrm{id}_Q$ for each $x \in Q$. For example, all core quandles are involutary. A quandle (resp. rack) Q is called commutative if $x * y = y * x$ for all $x, y \in Q$. . The dihedral quandle R_3 is commutative and no trivial quandle with more than one element is commutative.

3. Rota-Baxter operators on racks and quandles

3.1. Definition and simple properties.

Definition 3.1. (V. G. Bardakov) Let $(X, *)$ be a rack. A map $B: X \to X$ is said to be a Rota $-$ Baxter operator if the following identity holds:

$B(u) * B(v) = B((u * B(v)) * v)$, $\quad (u, v \in X)$

The triple $(X, *, B)$ is called a Rota $-$ Baxter rack.

For example, let $(X, *)$ be a quandle and $p \in X$ be a fixed element. The map $B(x) = p$ for any $x \in X$ is a Rota--Baxter operator.

Further, from the definition follows

Proposition 3.2 Let T be a trivial quandle. Any map $B: T \to T$ is a Rota--Baxter operator on T.

Proposition 3.3 If $B: X \to X$ is a RB-operator on a rack X then its image is a subrack.

We can define an algebraic operation $\circ: X \to X$ by the rule

$$u \circ v = (u * B(v)) * v \qquad u, v \in X$$

and formulate the next question:

Under which conditions this operation $\circ$ defines a rack (quandle) operation on X ?

An answer on this question gives

Proposition 3.4 Let $(X, *)$ be a quandle. Additionally, assume that

$a)$ for any $a, b \in X$ there exists unique $x \in X$ such that $(a * B(x)) * x = b$

$b)$ for any $u, v, w \in X$ holds

$$((u * B(v)) * B(w)) * w = ((u * B(w)) * w) * B((v * B(w)) * w)$$

Then $(X, \circ)$ is a rack.

Moreover, if the equality $u * (u * B(u)) = u$ holds for any $u \in X$, then $(X, \circ)$ is a quandle.

3.2 Rota--Baxter operators on dihedral quandle

1) Let $R_2 = \{a_0, a_1\}$ be the 2-element dihedral quandle that is a quandle with the multiplication

$a_o * a_0 = a_0$, $a_o * a_1 = a_0$, $a_1 * a_0 = a_1$, $a_1 * a_1 = a_1$,

We see that it is a trivial quandle. From Proposition 3.2 follows that any map $B: R_2 \to R_2$ is a Rota--Baxter operator.

2) Let $R_3 = \{a_0, a_1, a_2\}$ be the 3-element dihedral quandle. Let $B: R_3 \to R_3$ be a map,

$B(a_0) = a_{i_0}$, $B(a_1) = a_{i_1}$, $B(a_2) = a_{i_2}$.

In this case we will write $B = B_{\{i_o, i_1, i_2\}}$. Also, we will denote $|B| = |B_{\{i_o, i_1, i_2\}}|$ is the cardinality of the set .

As follows from Proposition if $i_o = i_1 = i_2$, then B is a RB-operator on R_3 . Hence, if $|B| = 1$ then it is a RB-operator. Suppose that $|B| = 2$. In this case we have 18 operators of this type:

$B_{\{011\}}$, $B_{\{022\}}$, $B_{\{100\}}$, $B_{\{122\}}$, $B_{\{200\}}$, $B_{\{211\}}$, $B_{\{101\}}$, $B_{\{202\}}$, $B_{\{010\}}$

$B_{\{212\}}$, $B_{\{020\}}$, $B_{\{121\}}$, $B_{\{110\}}$, $B_{\{220\}}$, $B_{\{001\}}$, $B_{\{221\}}$, $B_{\{002\}}$, $B_{\{112\}}$

If $|B| = 3$ then B is a permutation of R_3 and we have 6 such operators. It is need to check which from these maps are RB-operators.

Let as consider $B = B_{011}$. If it is a RB-operator, then

$B(a_0) * B(a_1) = B((a_0 * B(a_1)) * a_1)$

It is easy to see that the left hand side is equal to a_2, a_2 but the right hand side is equal to a_0 Hence, is not a RB-operator.

It is need to check all other operators.

Let as consider $B = B_{\{120\}}$ In this case $|B| = 3$. If it is a RB-operator, then $B(a_0) * B(a_0) = B((a_0 * B(a_0)) * a_0)$ It is easy to see that the left hand side is equal to a a_1 but the right hand side is equal to a_2. Hence, $B_{\{120\}}$ is not a RB-operator.

It is need to check all other maps $B = B_{\{i_o, i_1, i_2\}}$.

Proposition 3.5

Let $n \geq 3$ and $B: R_n \to R_n$ --mapping, Then

1) If $|B| = 1$, the mapping B is Rota-Baxter operator on R_n

2) If $|B| = 2$ then the mapping B is a Rota-Baxter operator on R_n if and only if n is even and for some k such that $0 \leq k \leq n/2 - 1$ the mapping B can be defined by one of the following formulas:

$$B(a_s)=\begin{cases} a_k, & s=2i, \\ a_{\frac{n}{2}+k}, & s=2i+1, \end{cases} \qquad B(a_s)=\begin{cases} a_{\frac{n}{2}+k}, & s=2i, \\ a_k, & s=2i+1, \end{cases}$$

for all $0 \le s \le n-1$

Proposition 3.6

Let p be an odd prime number. The mapping $B: R_p \to R_p$ is a Rota--Baxter operator if and only if .

Hypothesis

Let $n \ge 3$ and $B: R_n \to R_n$. If $\le 3 \le |\operatorname{Im} B| \le n$, then the mapping B is not a Rota-Baxter operator on R_n .

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