BASIC PROPERTIES OF A SLOWLY VARIABLE FUNCTION CHARACTERISTICS

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Abstract (English):
This article studies the properties of regular and slowly varying functions in the sense of Karamata. The regular variation of a function is a one-sided local asymptotic property of a given function. The class of functions with such properties arose as a result of attempts to extend the class of functions with level asymptotics around a point to the class of functions with “slowly varying” coefficients rather than level functions. The properties of slowly varying functions are used in number theory, in the theory of functions of complex variables, in the applications of probability theory and mathematical statistics.

Keywords: Random variable, distribution function, slowly varying function, regularly varying function, regularly varying order.

Introduction

The term regularity was first proposed in 1930 by the famous mathematician I. Karamata. However, the ideological foundations of this concept can be found in the works of E. Landau in 1911, and Polia before 1917. I. Karamata successfully applied his new theory to Tauber theorems (the Hardy-Littlewood-Karamata theorem dates back to this period). His ideas were developed by Karamata himself and his colleagues and students during the 1930s-1960s. The great possibilities for applying this theory to probability theory and its applications were realized by V. Feller in his book [1971], which aroused great interest in this topic. Another important impetus, from the point of view of probability, was given by L. de Haan.

E. Seneta systematically outlined the basic theory of this topic in his 1976 monograph. The Russian translation of this monograph, with additions, is a complete and easily accessible source of information for readers on the properties of functions of a regular variable.

The importance of regular variable functions was fully realized by probabilists, especially after the publication in 1966 of the second volume of V. Feller's book "Introduction to Probability Theory and Its Applications", which included elements of Karamata theory.

A regular variation of a function is a one-sided local asymptotic property of a given function, the choice of which is driven by the requirement to logically and conveniently expand the class of functions asymptotically near a given point.

A regular transformation, which is a local property, is defined with respect to a specific point. We assume the following.

Definition 1. A positive Λ function *at infinity* A function is called $[A,\infty)$, $A > 0$ *a regular variable if it* varies on a half-interval and such a number is found that, $\rho \in (-\infty;\infty)$ for an arbitrary number, $\lambda > 0$

$$\lim_{x\to\infty}\frac{\Lambda(\lambda x)}{\Lambda(x)} = \lambda^{\rho} \quad (1)$$

if equality is appropriate.
In this case, ρ the number Λ is called *the order of the function* .

If $\Lambda\left(\frac{1}{x}\right)$ it varies regularly at infinity, then $\Lambda(x)$ the function is said to be *regularly variable at zero* .

The concept of regular variation can now be defined by moving the origin of coordinates to an arbitrary finite point. Thus, we need only confine ourselves to constructing a theory of functions that vary regularly at infinity, and in the following we will drop the word "at infinity".
ρ a regular function of order $x^{\rho}L(x)$ in the form of . In this case, $L(x)$ - $[A,\infty)$ a function that varies positively on the interval, from relation (1) it follows that for an arbitrary $\lambda > 0$ number

$$\lim_{x\to\infty}\frac{L(\lambda x)}{L(x)} = 1 \quad (2)$$

appropriate.
So, $L(x)$ the function $\rho = 0$ is a function with an ordered regular variable.

Definition 2. A $\rho = 0$ function of regular variation with order is called $L(x)$ *a slowly varying function* .
The notation usually used for such functions $L(x)$ is derived from the first letter of the French word **lentement (slowly), and is related to the fact that the fundamental works on the theory of slowly varying functions were written by Karamata.**
So, a function $\Lambda(x)$ is a regular function $L(x)$ if and only if it is written in the form, in which case $x^{\rho}L(x)$ the function is a slowly varying function, $\rho \in (-\infty;\infty)$. We will use the latter view from now on.
$x \to \infty$ Not every function with a positive limit in , which is positive everywhere, is a slowly varying function. The simplest non-trivial example of a slowly varying function $\log x$ is the function , and furthermore, taking any multiple of its logarithm (for example, $\log\log x$) also leads to a slowly varying function.

BASIC THEOREMS
There are two main theorems in this theory concerning the properties of slowly varying functions. Each of them can be called fundamental in the sense that they are easily derived from each other, and most of the other properties of slowly varying functions follow from these theorems.

Theorem 1 (on smooth approximation). If $L(x)$ is a slowly varying function, then for any fixed $[a,b]$ $0 < a < b < \infty$ interval the relation (2) $\lambda \in [a,b]$ is equally valid with respect to .

Theorem 2 (about appearance). If $L(x)$ the function $[A,\infty)$, $A > 0$ is a slowly varying function defined on the semi-axis, then $B \geq A$ there is a number such that $x \geq B$ for all

$$L(x) = \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}dt\right\} \tag{3}$$

where $\eta(x)$ the function $[B,\infty)$ is a finite-dimensional function on the interval , and is continuous on $\eta(x) \to c(|c| < \infty)$, $\varepsilon(x) - [B,\infty)$ and $x \to \infty$ satisfies $\varepsilon(x) \to 0$ the condition .

By proving several lemmas, we first prove Theorem 1 and then Theorem 2.

In the following lemmas, it will be more convenient for us to work with $L(x)$ a function rather than $f(x) = \log L(e^x)$ a function .

So we $[\gamma,\infty)$ have a semi-axis that changes and

$$f(x+\mu) - f(x) \to 0,\ x \to \infty \tag{4}$$

of a real variable for $f(x)$ an arbitrary fixed point satisfying the condition μ.

Lemma 1. **The relation (4)** holds uniformly for all s in an arbitrarily fixed finite closed interval .μ

Proof. First, we prove Lemma 1 for a given $[0,1]$ segment μ. Let the lemma not be valid on this interval. Then there are $\varepsilon > 0$ numbers and $\{x_n\}, x_n \to \infty$ and $\{\mu_n\}$, $\mu_n \in [0,1]$ sequences such that n for all

$$|f(x_n+\mu_n) - f(x_n)| \geq \varepsilon \tag{5}$$

will be done.

$\{U_n\}$ and $\{V_n\}$ sets

$$U_n = \left\{\mu : \mu \in [0,2]\ |f(x_m+\mu) - f(x_m)| < \frac{1}{2}\varepsilon,\ \forall m \geq n\right\} \tag{6}$$

$$V_n = \left\{\lambda : \lambda \in [0,2]\ |f(x_m+\mu_m+\lambda) - f(x_m+\mu_m)| < \frac{1}{2}\varepsilon,\ \forall m \geq n\right\} \tag{7}$$

we determine through relationships.

Obviously, the sets $\{U_n\}$ and $\{V_n\}$ are dimensional, and by (4) each $[0,2]$ da $U_n, V_n - 0$ of the sets is a set of monotonically increasing sequences due to .

Therefore, $m(\cdot)$ if denotes the size of the set, then N for arbitrary large enough s $m(U_n) > \frac{3}{2}$, $m(V_n) > \frac{3}{2}$. $V_n' = V_n + \mu_n$ let be, then $m(V_n') = m(V_n) > \frac{3}{2}$. Note that this is $m(U_n) \subset [0,2] \subset [0,3]$ $V_n' \subset [0,3]$.

Hence, $U_n \cap V_n' = \emptyset$ (here $\emptyset$ - the empty set), $\mu \in U_n$ exists, and for $\mu - \mu_N \in V_N$, so from (6)

$$|f(x_N + \mu) - f(x_N)| < \frac{1}{2}\varepsilon \qquad (8)$$

From (7)

$$|f(x_N + \mu_N + \mu - \mu_N) - f(x_N + \mu_N)| < \frac{1}{2}\varepsilon \qquad (9)$$

or equivalent

$$|f(x_N + \mu) - f(x_N + \mu_N)| < \frac{1}{2}\varepsilon$$

comes from.

Using relations (8) and (9) and the triangle inequality,

$$|f(x_N + \mu) - f(x_N)| < \frac{1}{2}\varepsilon$$

we get the inequality, but this is the opposite of (5).

an arbitrary $[a,b]$ $b > a$ segment, we obtain the function $\tilde{f}(x) = f((b-a)x)$ by the formula $\tilde{f}$. Hence,

$$f(x+\mu) - f(x) = \tilde{f}(y+\nu) - \tilde{f}(y) + f(x-a) - f(x),$$

here $y = \frac{x-a}{b-a}$, $\nu = \frac{\mu - a}{b-a}$.

$y \to \infty \Leftrightarrow x \to \infty$; $\mu \in [a,b] \Leftrightarrow \nu \in [0,1]$

The lemma has been proven.

Lemma 2. f For a function X, $X \geq \gamma$, a number is found such that f the function is bounded on every $[X, X']$ $X' \geq X$ interval.

Proof. By Lemma 1, there exists a number $|f(x+\mu) - f(x)| < 1$, $x > X$, $\forall \mu \in [0,1]$ for X.

we assume that , we have $x = X$, $X + \mu = y$ for $|f(y)| \leq |f(x)| + 1$ each $y \in [X, X+1]$.

this for each $x \in [X+1, X+2]$,

$$|f(x)| \leq |f(x) + 1| + 1 \leq |f(x)| + 2$$

We will get the grade.

for $[X+k-1, X+k]$ every natural number the inequality is valid k in the section $|f(x)| \le |f(X)| + k$, so $[X, X+k]$ it is also valid in the section.

The lemma has been proven.

Result : f The function is integrable on an arbitrary $X' > X$ interval $[X, X']$ (as a measurable, bounded function on this interval).

Lemma(3). If X If the conditions of Lemma 2 are satisfied, then $x \ge X$ for

$$f(x) = \chi(x) + \int_X^x \varepsilon(t)dt$$

is appropriate, here χ *va* ε - $[X, X']$, $X > X'$ bounded and measurable in an arbitrary cross-section, moreover $\chi(x) \to c$ $(|c| < \infty)$ and $\varepsilon(x) \to 0,\ x \to \infty$.

Proof. Applying Lemma 2, $x \ge X$ for

$$f(x) = \int_x^{x+1}\big(f(x) - f(t)\big)dt + \int_X^x \big(f(t+1) - f(t)\big)dt + \int_X^{X+1} f(t)dt.$$

The sum of the three integrals on the right-hand side can be written as follows, respectively:

$$\delta(x) + \int_X^x \varepsilon(t)dt + c.$$

From relation (4) it follows $\varepsilon(t) = f(t+1) - f(t) \to 0,\ t \to \infty$ that , and from Lemma 1

$\delta(x) = \int_x^{x+1}\big(f(x) - f(t)\big)dt = \int_0^1 \big(f(x) - f(x+\mu)\big)d\mu,\ x \to \infty$ it follows that .

$\chi(x) = \delta(x) + c$ If we put , the lemma is proved.

Lemma 4. It is found that X, $X \ge X^*$ all $x \ge X^*$ for the

$$f(x) = \chi^*(x) + \int_{X^*}^x \varepsilon^*(t)dt \tag{10}$$

is reasonable, where the functions χ^* and ε^* have all the properties of Lemma 3, and furthermore, ε^* the function is continuous.

Proof. $f^*(x) = \int_X^x \varepsilon(t)dt = \int_X^x \big(f(t+1) - f(t)\big)dt$, then

$$\chi(x) = f(x) - f^*(x) \to c,\ x \to \infty \tag{11}$$

If $\mu > 0$ we fix , this equality holds:

$f^*(x+\mu) - f^*(x) = \int_x^{x+\mu} \left(f(t+1) - f(t)\right)dt = \int_0^{\mu} \left(f(y+x+1) - f(y+x)\right)dy$.

Now $y \in [0, \mu]$ at

$f(y+x+1) - f(y+x) = f(y+x+1) - f(x) - \left(f(y+x) - f(y)\right)$ and by Lemma 1 this expression $y \in [0, \mu]$ tends to zero uniformly at , when $f^*(x+\mu) - f^*(x) \to 0, \ x \to \infty$.

The last relation is true for any $\mu > 0$, its truth $\mu < 0$ can be proven in the same way for , and $\mu = 0$ is easily verified for . Hence, it is μ true for all real .

We can see that Lemmas 1–3 f^* can be applied to the function with X some $X^*, \ X^* \geq X$ replacement, i.e.

$$f^*(x) = \delta^*(x) + \int_{X^*}^{x} \varepsilon^*(t)dt + c^*$$

this on the ground $\varepsilon^*(t) = f^*(t+1) - f^*(t)$ function f^* continuity because of is continuous . Then from (11)

$$f(x) = c(x) + f^*(x) = c(x) + \delta^*(x) + \int_{X^*}^{x} \varepsilon^*(t)dt + c^*$$

If we take , we get the desired result.

Note. By repeating the procedure described in the proof of this lemma as many times as necessary, we obtain the expression (10) We can obtain a representation of $\varepsilon^*(t)$ the function t with any fixed order derivative for sufficiently large values of . $f(x)$ All the " unpleasant properties" of the function $c^*(x)$ are collected in the function, which we can only say about $x \to \infty$ when it has a finite limit.

$f(x) = \log L(e^x)$ we get the inverse substitution. Then $x > 0$ for any $L(x) = \exp\{f(\log x)\}$.

Now Theorems 1 and 2 follow from Lemmas 1 and 4, respectively. So, in the integral representation theorem $\eta(x) = c^*(\log x), \ \varepsilon(x) = \varepsilon^*(\log x)$ we can get , because

$\int_{X^*}^{\log x} \varepsilon^*(t)dt = \int_B^x \frac{\varepsilon^*(\log y)}{y}dy$, here $B = \exp X^*$.

Theorem 2 results . From Theorem 2, the arbitrary expression (3) is $\eta(x)$ *va* $\varepsilon(x)$ functions are slowly varying functions.

It is worth noting that one consequence of expression (3) is that x for sufficiently large , we can write a slowly varying function $L(x) = M(x)L_0(x)$ in the form , where $M(x)$ the function is positive, measurable, and bounded on intervals sufficiently far from the origin of coordinates, and $x \to \infty$ tends to $L_0(x)$ a positive limit at , where M and the function is a particularly "good" slowly varying function. So, $L(x) \sim ML_0(x),\ x \to \infty$, where

$$L_0(x) = \exp\left(\int_B^x \frac{\varepsilon(t)}{t}\,dt\right)$$

$\varepsilon(t)$ and the function is continuous and $t \to \infty$ tends to zero at . In fact, we have obtained the form and $\varepsilon(t)$ for the function itself

$$\varepsilon(t) = \frac{tL_0'(t)}{L_0(t)}, \tag{12}$$

b is a die, where the bar symbol denotes the derivative. For the last relation, we can formulate an elementary but important counter-relation: for an arbitrary positive, $x \geq B,\ B > 0$ has a continuous derivative at,

$$\frac{xg'(x)}{g(x)} \to 0,\ x \to \infty \tag{13}$$

that satisfies the condition $g(x)$ is a slow variable. To check this, we denote the function on the left side of (13) by , find $\varepsilon(x)$ the expression $\varepsilon(x)$ by integrating, $g(x)$ and use the above conclusion. If the number on the right side of (13) $\rho,\ \rho \in (-\infty,\infty)$ is , then it is not difficult to see that $g(x)$ function ρ - is a function of a regular variable of order.

and conditions $\lim\limits_{x\to\infty}\frac{L(\lambda x)}{L(x)} = 1$ that define the concepts of regular and slow change $\lim\limits_{x\to\infty}\frac{\Lambda(\lambda x)}{\Lambda(x)} = \lambda^{\rho}$ can be significantly weakened without harming the developed theory.

Let us be given the following auxiliary result:

Lemma 5. Λ The function *is positive* , $[A,\infty), A > 0$ *varies on the semi-axis and* $\lambda \in [a,b],\ 0 < a < b < \infty$ *is*

$$\lim_{x\to\infty}\frac{\Lambda(\lambda x)}{\Lambda(x)} = \varphi(\infty) \tag{14}$$

Let be a function satisfying the condition, where is $[a,b]$ *fixed,* $\varphi(\lambda)$ *and the function in question is positive and bounded. In this case, all* $\lambda > 0$ *for which the relation (14) is valid, the positive bounded* $\varphi(\lambda)$ *function is found.*

Proof. Let us take a number satisfying $\gamma > 0$ a condition $a \le \frac{\lambda}{\gamma} \le b$; then $\lambda \in [a,b]$ for an arbitrary fixed

$$\frac{\Lambda(\lambda x)}{\Lambda(x)} = \frac{\Lambda\left(\gamma\left(\frac{\lambda x}{\gamma}\right)\right)}{\Lambda\left(\frac{\lambda x}{\gamma}\right)} \frac{\Lambda\left(\frac{\lambda x}{\gamma}\right)}{\Lambda(x)}$$

from view and its limit exists since it is ($\varphi(\gamma)$ we define it through)

$$\lim_{x\to\infty} \frac{\Lambda(\gamma x)}{\Lambda(x)} = \lim_{x\to\infty} \frac{\Lambda\left(\gamma\left(\frac{\lambda x}{\gamma}\right)\right)}{\Lambda\left(\frac{\lambda x}{\gamma}\right)} = \frac{\varphi(\lambda)}{\varphi\left(\frac{\lambda}{\gamma}\right)} > 0$$

we get.

This limit $\gamma \le \frac{\lambda}{a} \le \frac{b}{a}$ exists for $\gamma \in \left[\frac{a}{b}, \frac{b}{a}\right]$ and , for which $\gamma \ge \frac{\lambda}{b} \ge \frac{a}{b}$ also

$$\frac{\Lambda(\gamma x)}{\Lambda(x)} \to \varphi(\gamma) \ge 0,\ x \to \infty.$$

If we repeat this argument $k-1$ once, $\gamma \in \left[\left(\frac{a}{b}\right)^k, \left(\frac{b}{k}\right)^k\right]$

$$\frac{\Lambda(\gamma x)}{\Lambda(x)} \to \varphi(\gamma) > 0,\ x \to \infty$$

what we will get $\frac{a}{b} < 1, \frac{b}{a} > 1$ Since k, by choosing , the above relationship $\gamma > 0$ can be obtained for any .

The lemma has been proven.

Lemma 6. Let the conditions of Lemma 5 *be satisfied, and in addition,* $\lambda > 0, \lambda \in S$ *let equality (14) be satisfied, where is* S *a positive-dimensional set,* $\varphi(x)$ *and the function is positive and finite. Then Lemma 1 remains valid* .

Proof.

$$f(x) = \log\Lambda\left(e^x\right),\ \ x > 0,$$

$$\psi(\tau) = \log\varphi\left(e^\tau\right),\ \ \tau \in S^*,\ \ S^* = \left\{\tau : e^\tau \in S\right\}.$$

we get functions.

Then $x \to \infty,\ \tau \in S^*$ at

$$f(x+\tau)-f(x)\to\psi(\tau) \tag{15}$$

Later, $\nu\in S^*$ at

$$f(x+\tau+\nu)-f(x)=f(x+\tau+\nu)-f(x+\tau)+$$
$$+f(x+\tau)-f(x)\to\psi(\nu)+\psi(\tau), \qquad x\to\infty$$

therefore (15) $\mu\in D=\{\mu;\mu=\tau+\nu,\tau,\nu\in S^*\}$ is also valid. Since is S^* a positive dimensional set, according to a certain theorem, the set defined in this way D is somehow I contains a closed interval. Hence, $\mu\in I$ for an arbitrary

$$f(x+\mu)-f(x)\to\psi(\mu)$$

$\psi(\mu)$ (if necessary) $\psi(\nu)+\psi(\tau)$, where $\mu=\nu+\tau,\nu,\tau\in S^*$. f and ψ taking the inverse substitutions, we arrive at the conditions of Lemma 1. The lemma is proved.

The main theorem of this topic is the following theorem, which shows that $\varphi(\lambda)$ a function λ^ρ has the form , that is, Λ the function varies regularly in the real sense.

Theorem 3 . *(Theorem on Characteristics). Under the condition of Lemma 6, there is $\varphi(\lambda)$ a function λ^ρ, where $\rho\in(-\infty,\infty)$.*

Proof . By Lemma 6, it is optional $\lambda>0$.

$$\lim_{x\to\infty}\frac{\Lambda(\lambda x)}{\Lambda(x)}=\varphi(\lambda)>0$$

Then optional $\gamma>0$ for

$$\frac{\Lambda(\lambda\gamma x)}{\Lambda(\gamma x)}\cdot\frac{\Lambda(\gamma x)}{\Lambda(x)}=\frac{\Lambda(\lambda\gamma x)}{\Lambda(x)}$$

will be, so $x\to\infty$ at

$$\varphi(\lambda)\varphi(\gamma)=\varphi(\lambda\gamma)\ \ \forall\lambda,\gamma>0, \tag{16}$$

we get.

The last relation $\varphi>0$ is the Gamel equation for the function in positive numbers, which is the limit point of the u-dimensional functions, and is itself dimensional. It is known that under this condition the solution of (16) λ^ρ, $\rho\in(-\infty,\infty)$ can only have the form. The theorem is proved.

However, it is very useful to give a simple, direct proof of the last fact, since this is often not done in elementary textbooks. The given proof also serves as an illustration of the application of Luzin's theorem to this problem. This theorem, along with the theorems of Yegorov and Steinhaus, serves as a natural tool of measure theory, which is used in the theory we present, as a number of authors have pointed out.

Theorem 3.2. *If a positive dimensional bounded $\varphi(\lambda)$, $\lambda>0$ function satisfies the conditions of (16), it λ^ρ has the form, where $\rho\in(-\infty,\infty)$.*

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