NOVATEUR PUBLICATIONS
JournalNX- A Multidisciplinary Peer Reviewed Journal
ISSN No: 2581 - 4230
VOLUME 11, ISSUE 12, December – 2025

CHARACTERISTIC FUNCTIONS OF RANDOM VARIABLES AND THEIR PROPERTIES

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Abstract

In probability theory, methods and analytical apparatus from various branches of mathematical analysis are widely used. Simple solutions to many problems encountered in probability theory, especially those involving sums of uncorrelated random variables, can be found using characteristic functions, developed in mathematical analysis and known as Fourier transforms. The fact that the method of characteristic functions is one of the main tools of the analytical apparatus of probability theory can be clearly seen in the proof of limit theorems, in particular, in the proof of the central limit theorem, which generalizes the Moivre-Laplace theorem. In this article, we will limit ourselves to describing the main properties of characteristic functions.

Keywords: Random variable, distribution function, characteristic function, density function.

Annotatsiya

Ehtimollar nazariyasida matematik analizdagi turli boʻlimlarning metodlari va analitik apparatlari keng qoʻllaniladi. Ehtimollar nazariyasida uchraydigan juda koʻp masalalarning ayniqsa oʻzaro bogʻliqsiz tasodifiy miqdorlarning yigʻindisi bilan bogʻliq boʻlgan masalalarning sodda yechimlarini matematik analizda rivojlantirilgan va Furye almashtirishlari nomi bilan ma’lum boʻlgan xarakteristik funksiyalar yordamida topish mumkin. Xarakteristik funksiyalar metodi ehtimollar nazariyasi analitik apparatining asosiy vositalaridan biri ekanligini limit teoremalarni isbotlashda, xususan Muavr-Laplas teoremasini umumlashtiruvchi markaziy limit teoremani isbotlash jarayonida yaqqol koʻrishimiz mumkin. Ushbu maqolada biz xarakteristik funksiyalarning asosiy xossalarini bayon qilish bilan chegaralanamiz.

Аннотация(Russian):

В теории вероятностей широко используются методы и аналитический аппарат из различных разделов математического анализа. Простые решения многих задач, встречающихся в теории вероятностей, особенно тех, которые связаны с суммами некоррелированных случайных величин, можно найти с помощью характеристических функций, разработанных в математическом анализе и известных как преобразования Фурье. Тот факт, что метод характеристических функций является одним из основных инструментов аналитического аппарата теории вероятностей, ясно виден в доказательстве предельных теорем, в частности, в доказательстве центральной предельной теоремы, которая обобщает теорему Муавра-Лапласа. В этой статье мы ограничимся описанием основных свойств характеристических функций

The analytical methods of probability theory are based on the functional properties of mathematical tools such as characteristic functions and generating functions. These functions are also considered important instruments in mathematical analysis. In particular, characteristic functions can be regarded as a generalization of the well-known Fourier transforms.

Let, $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space, where:

- Ω - the sample space,
- $\mathcal{F}$ – a sigma-algebra defined on this space,
- P – a probability measure. A measurable function that elements of Ω the sample space into the set $\mathbf{R}$, $\xi = \xi(\omega)$ is called a random variable. . In other words, the preimage

$$\xi^{-1}(B) = \{\omega : \xi(\omega) \in B\} \in \mathfrak{F}.$$

here , $\omega \in \Omega$ and B – Borel set or a sigma – algebra defined on the set $\mathbf{R}$ (the system of Borel sets $\mathfrak{B} = \mathfrak{B}(\mathbf{R})$) that is $B \in \mathfrak{B}(\mathbf{R})$.

In this case , the random variable ξ is said to map the space $(\Omega, \mathfrak{F})$ into the measurable space $(\mathbf{R}, \mathfrak{B})$: $\xi : (\Omega, \mathfrak{F}) \longrightarrow (\mathbf{R}, \mathfrak{B})$. By means of this mapping , a probability measure $\mathbf{P}_\xi$ on the measurable space $(\mathbf{R}, \mathfrak{B})$ is defined as follows: $\mathbf{P}_\xi\{B\} = \mathbf{P}\{\xi \in B\}$

Thus, we obtain a new measurable space $(\mathbf{R}, \mathfrak{B}, \mathbf{P}_\xi)$. The probability value $\mathbf{P}_\xi\{B\}$ defined above is called the distribution of the random variable ξ . in particular, if we choose $B = (-\infty; x)$ then the function defined on $\mathbb{R}$

$$F_\xi(x) \coloneqq \mathbf{P}_\xi\{\xi \in (-\infty; x)\} = \mathbf{P}\{\xi < x\}$$

is called the *distribution function* of the random variable ξ.

It is clear that the distribution function of a random variable uniquely determines its distribution.

Definition 1. Let $\xi = \xi(\omega)$ be a random variable defined *mathematical expectation* on the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$

$$\int_\Omega \xi(\omega)\mathbf{P}(d\omega)$$

The *mathematical expectation* or *mean value* of the random variable is defined as:

$$\mathbf{M}\xi = \int_{\mathbf{R}} x\mathbf{P}_\xi(dx) = \int_{\mathbf{R}} x dF(x)$$

which can be written in the form above, where $F(x) = F_\xi(x)$.

From this definition, it is clear that if $\mathbf{M}|\xi| < \infty$ then , $\mathbf{M}\xi < \infty$. For example, if $1 - F_{\xi}(x) > 1/x$, $x \to \infty$ then the random variable $\xi = \xi(\omega)$ does not have a finite mathematical expectation.In the process of sufficiently studying the theory of random variables, together with real-valued random variables $\xi = \xi(\omega)$, the concept of *complex-valued random variables* may also be introduced. By general probabilistic theory, we consider a complex random variable as the sum $\xi = \xi_1(\omega) + i\xi_2(\omega)$, where $(\xi_1;\xi_2)$ is a random vector. Naturally, the mathematical expectation of such a random variable is $\mathbf{M}\xi = \mathbf{M}[\xi_1(\omega) + i\xi_2(\omega)] = \mathbf{M}\xi_1 + i\mathbf{M}\xi_2$. Two complex-valued random variables $\xi = \xi_1(\omega) + i\xi_2(\omega)$ va $\eta = \eta_1(\omega) + i\eta_2(\omega)$ are called *independent*, if the random vectors $(\xi_1;\xi_2)$ and $(\eta_1;\eta_2)$ generate independent $\sigma(\xi_1;\xi_2)$ and $\sigma(\eta_1;\eta_2)$ sigma-algebras. For such variables, verifying the identity

$$\mathbf{M}\xi\eta = \mathbf{M}\xi \cdot \mathbf{M}\eta$$

is straightforward.

Definition 2. Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space. The *characteristic function* of a real-valued random variable $\xi = \xi(\omega)$ is the complex-valued function:

$$\varphi_{\xi}(t) = \mathbf{M}e^{it\xi} = \int_{\mathbf{R}} e^{itx} dF_{\xi}(x), \quad t \in \mathbf{R}.$$

If the distribution function $F(x) = F_{\xi}(x)$ has a density $f(x) = F'(x)$, then the characteristic function takes the form:

$$\varphi_{\xi}(t) = \mathbf{M}e^{it\xi} = \int_{\mathbf{R}} e^{itx} f(x)dx$$

As can be seen, this function is the Fourier transform of the function $f(x)$ known from mathematical analysis. In the general case, however, the characteristic function $F(x)$ is the Fourier–Stieltjes transform applied to the distribution function. For any random variable, the characteristic function always exists. This fact is confirmed by the following inequality, which holds for all $t \in \mathbf{R}$:

$$|\varphi_{\xi}(t)| = \left|\int_{\mathbf{R}} e^{itx} dF(x)\right| \le \int_{\mathbf{R}} |e^{itx}| dF(x) \le \int_{\mathbf{R}} dF(x) = 1.$$

This follows from the Euler formula known from complex analysis:

$$|e^{itx}| = |\cos tx + i \sin tx| = \sqrt{\cos^2 tx + \sin^2 tx} = 1.$$

And from the property of the distribution function: $\int_{-\infty}^{\infty} dF(x) = 1$.

Characteristic functions serve as an excellent mathematical tool in studying sums of independent random variables. We now focus on exactly these aspects.

We begin by studying the main properties of characteristic functions:

1°. For any random variable:

$$\varphi_{\xi}(0) = 1.$$

2°. If a random variable η is obtained by a linear transformation of ξ, that is: $\eta = a\xi + b,\ a,b \in \mathbf{R}$, then:

$$\varphi_{\eta}(t) = e^{itb}\varphi_{\xi}(at).$$

3°. If $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables, then the characteristic function of their sum $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ is

$$\varphi_{S_n}(t) = \varphi_{\xi_1}(t)\cdot\varphi_{\xi_2}(t)\cdot\ldots\cdot\varphi_{\xi_n}(t).$$

4°. The characteristic function is uniformly continuous.
5°. If a random variable ξ has a finite k-th moment, that is:

$$\mathbf{M}|\xi|^k < \infty,\ k \in \mathbf{N}$$

then its characteristic function $\varphi_{\xi}(t)$ has a continuous derivative of order k, and:

$$\mathbf{M}\xi^k = \frac{\varphi_{\xi}^{(k)}(0)}{i^k}.$$

6°. The function $\varphi(t)$ is a real-valued function if and only if the distribution function $F(x)$ is symmetric,

$$\left[\int_B dF(x) = \int_{-B} dF(x)\right] \Leftrightarrow \left[\varphi(t) \in \mathbf{R}\right] \quad B \in \mathcal{B}(\mathbf{R}),\ -B = \left\{-x : x \in B\right\}.$$

Now, using the properties of characteristic functions provided above, we compute the characteristic functions of some distributions frequently used in probability theory.

1. "Unit" distribution. Recall that a random variable x is said to have a *unit (degenerate)* distribution if it takes only one fixed value a with probability 1, i.e., $P(x = a) = 1$. In this case,

$$\varphi_x(t) = Me^{itx} = e^{iat}.$$

2. Bernoulli distribution. Let the random variable ξ take values

$$x = \begin{cases} 1, & ehtimolligi \quad p, \\ 0, & ehtimolligi \quad 1-p. \end{cases}$$

Then the characteristic function is

$$\varphi_x(t) = Me^{itx} = e^{it}\cdot p + e^{it\cdot 0}(1-p) = pe^{it} + 1 - p.$$

3. Binomial distribution. Let ξ take values $m = 0,1,\ldots,n$ with probabilities

$$P(x = m) = C_n^m p^m (1-p)^{n-m} \quad m = 0,1,\ldots,n$$

This random variable can be represented as $\xi = \xi_1 + \ldots + \xi_n$, where ξ_i are independent Bernoulli random variables with parameter p. Therefore, using independence,

$$\varphi_\xi(t) = Me^{it\xi} = Me^{it(\xi_1 + \ldots + \xi_n)} = Me^{it\xi_1} \cdot \ldots \cdot Me^{it\xi_n} =$$

$$= \left(pe^{it} + 1 - p\right)^n .$$

4. Poisson distribution. A random variable ξ_i takes values $k = 0,1,2,\ldots$

$$P(\xi = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad \lambda > 0,\ k = 0,1,2,\ldots$$

then,

$$\varphi_\xi(t) = Me^{it\xi} = \sum_{k=0}^{\infty} e^{itk} P(\xi = k) = \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\left(\lambda e^{it}\right)^k}{k!} = e^{-\lambda} \cdot e^{\lambda e^{it}} = e^{\lambda\left(e^{it} - 1\right)}.$$

5. Geometrik taqsimot.

$$P(\xi = n) = pq^{n-1}, \qquad n = 1,2,\ldots, \quad q = 1 - p$$

and

$$\varphi_\xi(t) = Me^{it\xi} = \sum_{n=1}^{\infty} e^{itn} pq^{n-1} = pe^{it} \sum_{n=1}^{\infty} \left(qe^{it}\right)^{n-1} = \frac{pe^{it}}{1 - qe^{it}}.$$

6. Normal distribution. A continuous random variable ξ is said to have a normal (or Gaussian) distribution if its density function is given by

$$f(x,a,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty ,$$

where $\left(a,\sigma^2\right)-$ are parameters, $a \in R, \quad \sigma > 0$.

Let us find the characteristic function of the standard normal distribution with parameters (0,1):

$$\varphi_\xi(t) = Me^{it\xi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx - \frac{x^2}{2}} dx \qquad (1).$$

Differentiating the equation (1) with respect to t, we get

$$\varphi'(t) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{itx - \frac{x^2}{2}} dx .$$

If we differentiate again and perform integration by parts, we obtain

$$\varphi'(t) = \frac{i}{\sqrt{2\pi}}\left[\left.-e^{itx-\frac{x^2}{2}}\right|_{-\infty}^{\infty} + it\int_{-\infty}^{\infty} e^{itx-\frac{x^2}{2}}dx\right] = -t\varphi(t).$$

Hence we obtain the relation (for the characteristic function of the standard normal distribution)

$$\varphi'(t) + t\varphi(t) = 0 \qquad (2)$$

With the initial condition , $\varphi(0) = 1$ the solution of this differential equation is

$$\varphi(t) = e^{-\frac{t^2}{2}} \qquad (3)$$

Thus we have found the characteristic function.

Now let us find the characteristic function of a normal distribution with parameters (a, σ^2). If we denote by ξ_c a standard normal random variable whose characteristic function is given by (3), then any normal random variable ξ with parameters (a, σ^2) can be written in the form

$$\xi = \sigma\xi_c + a.$$

Therefore

$$\varphi_{\xi}(t) = \varphi_{\sigma\xi_c + a}(t) = e^{iat}\cdot\varphi_{\xi_c}(t\sigma) = e^{iat - \frac{\sigma^2 t^2}{2}}.$$

7. Uniform distribution on $[a,b]$. In this case, the distribution is continuous, and its density function is

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

The corresponding characteristic function is

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \frac{1}{b-a}\int_a^b e^{itx}dx = \frac{e^{itb} - e^{ita}}{it(b-a)}$$

Let us recall some special cases:

1) If $a = -\lambda,\ b = \lambda$ then

$$\varphi(t) = \frac{e^{i\lambda t} - e^{-i\lambda t}}{2it\lambda} = \frac{\sin \lambda t}{\lambda t}.$$

2) $a = 0,\ b = L$ then

$$\varphi(t) = \frac{e^{itL} - 1}{itL}.$$

8. Gamma distribution. In this case, the density function is

$$f_a(x) = \frac{x^{a-1}}{\Gamma(a)} e^{-x}, \qquad x \geq 0.$$

Let us denote by $\varphi_a(t)$ the characteristic function corresponding to this density. Before proceeding, let us confirm the following fact:

It is easy to see that the density function, $f_{a+b}(x)$ is obtained as the composition of the functions $f_a(x)$ and $f_b(x)$. Indeed,

$$f_{a+b}(x) = \int_0^x f_b(x-u) f_a(u) du = \frac{e^{-x}}{\Gamma(a)\cdot\Gamma(b)} \int_0^x u^{a-1}(x-u)^{b-1} du$$

$$= \frac{x^{a+b-1}\cdot e^{-x}}{\Gamma(a)\cdot\Gamma(b)} \cdot \int_0^1 y^{a-1}(1-y)^{b-1} dy .$$

The last integral is known as Euler's $B(a,b)$-integral and is related to the Gamma function:

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

So, $f_{a+b}(x) = \dfrac{x^{a+b-1}}{\Gamma(a+b)} e^{-x}, \qquad x \geq 0$ equality is apprioprite. We note that $f_{a+b} = f_a * f_b$.

Let us first consider the case $a = 1$, then we count

$$\varphi_1(t) = \int_0^\infty e^{itx} f_1(x) dx .$$

We compute this integral. By integrating by parts, we obtain

$$\varphi_1(t) = -e^{itx-x}\Big|_0^\infty + it\int_0^\infty e^{itx-x} dx = 1 + it\varphi_1(t).$$

$$\varphi_1(t) = \frac{1}{1-it} \qquad (5)$$

Thus we find the expression (5).

For any $n \geq 1$, using (4) and (5), we obtain

$$\varphi_n(t) = \frac{1}{(1-it)^n}.$$

From this, it follows that

$$\varphi_1(t) = \left[\varphi_{\frac{1}{n}}(t)\right]^n, \quad \varphi_{\frac{1}{n}}(t) = (1 - it)^{-1/n},$$

$$\varphi_{\frac{m}{n}}(t) = \left[\varphi_{\frac{1}{n}}(t)\right]^m = (1 - it)^{-m/n}.$$

Thus we can write the following equalities. Hence for any rational α :

$$\varphi_\alpha(t) = (1 - it)^{-\alpha} \quad (6)$$

Therefore, (6) also holds for any real α . Since the density function $p_\alpha(x)$ is continuous with respect to the parameter α ,

$$f_{\alpha_n}(x) \to f_\alpha(x), \qquad \alpha_n \to \alpha,$$

and consequently ,

$$\varphi_{\alpha_n}(t) \to \varphi_\alpha(t).$$

Thus formula (6) holds for all $\alpha > 0$. If α is a rational number, then condition $\varphi_\alpha(0) = 1$ ensures that the limit is consistent.

In probability theory, in many applications it is important to find the distribution function corresponding to a given characteristic function. The following theorem shows that the characteristic function uniquely determines the distribution.

Theorem 1. (Uniqueness Theorem). *If two distribution functions $F(x)$ and $G(x)$ have the same characteristic function, i.e.*

$$\int_{\mathbf{R}} e^{itx} dF(x) = \int_{\mathbf{R}} e^{itx} dG(x),$$

then these functions are identical:

$$F(x) \equiv G(x).$$

The theorem stated above shows that the distribution function of a random variable can be recovered uniquely from its characteristic function.

Theorem 2. (Inversion Formula). *Let a distribution function $F(x)$ and its characteristic function be given. The following statements hold at every point a, b $(a < b)$, where $F = F(x)$ is continuous:*

$$F(b) - F(a) = \frac{1}{2\pi} \int_{\mathbf{R}} \varphi(t) \frac{e^{-ita} - e^{-itb}}{it} dt ;$$

If

$\int_{\mathbf{R}} |\varphi(t)| dt < \infty$ *,then $F = F(x)$ has a density function, and*

$$F(x) = \int_{-\infty}^{x} f(v)dv,$$

and moreover

$$f(x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-itx}\varphi(t)dt.$$

Let

$$x_1, x_2, \ldots, x_n, \ldots \qquad (7)$$

be a sequence of mutually independent and identically distributed random variables. Let

$$S_n = x_1 + \ldots + x_n.$$

From the course of probability theory, we know that the *Law of Large Numbers* holds for such a sequence, meaning the average $Mx_k = a$ exists. However, for the sequence (7), for the central limit theorem to hold, it is required that the variances of the independent random variables exist, that is, $Dx_n = \sigma^2$

Then the standardized and normalized random variable

$$\eta_n = \frac{S_n - MS_n}{\sqrt{DS_n}} = \frac{S_n - na}{\sigma\sqrt{n}}$$

will have the distribution function $F_n(x)$, and its limit distribution is the standard normal distribution with parameters (0,1), denoted by $\Phi(x)$.This means that

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du.$$

Levy's Theorem. *If* $0 < \sigma^2 < \infty$ *, then the convergence*

$$\sup_x \left|F_n(x) - \Phi(x)\right| \to 0, \quad n \to \infty,$$

holds, i.e., (7) becomes the central limit theorem.

In this case, the sequence $\{\eta_n, n \geq 1\}$ converges asymptotically to the normal distribution.

Proof. Without loss of generality, we may assume $a = 0$, because otherwise the sequence $\{x_n^1 = x_n - a, n \geq 1\}$ can be considered instead, and then the same results apply.

It can be shown that the characteristic function

$$\varphi_x(t) = Me^{it\eta_n} \to e^{-t^2/2}$$

is sufficient. Indeed, if $a = 0$, then

$$\varphi_{\eta_n}(t) = f^n\left(\frac{t}{\sigma\sqrt{n}}\right), \quad f(t) = Ee^{itx_1},,$$

and the existence of $M\xi_n^2$ ensures the existence of $|\varphi''(t)|$. Applying Taylor's formula, we get

$$\varphi(t)=\varphi(0)+\varphi'(0)t+\varphi''(0)\cdot\frac{t^2}{2}+o\left(t^2\right)=1-\frac{t^2\sigma^2}{2}+o\left(t^2\right)$$

Thus, as $n\to\infty$,

$$\ln\varphi_n(t)=n\ln\left[1-\frac{\sigma^2}{2}\cdot\left(\frac{t}{\sigma\sqrt{n}}\right)^2+o\left(\frac{t}{n}\right)^2\right]==n\left(-\frac{t^2}{2^n}+o\left(\frac{t^2}{n}\right)\right)=-\frac{t^2}{2}+o(1)\to-\frac{t^2}{2}.$$

The theorem has been proved.

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